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Rooted planar maps modulo some patterns

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ABSTRACT

We provide generating functions for the number of equivalence classes of rooted planar maps where two maps are equivalent whenever their representations in shuffles of Dyck words coincide on all occurrences of a given pattern.

1. Introduction and notations

A *map* is defined topologically as a 2-cell imbedding of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. A *rooted planar map* is a map on a sphere with a distinguished edge, called the *root*, assigned with an orientation. Fig. 1(a) shows a rooted planar map with seven edges. We refer to [1,15,16] for the enumeration of rooted planar maps with respect to the number of edges. Planar maps have been widely studied for their combinatorial structure and their links with other domains such as theoretical physics where they appear in some models of 2-dimensional quantum gravity for instance. From a combinatorial point of view, it is proved in [8] that rooted planar maps are in one-to-one correspondence with shuffles of Dyck words that avoid a specific pattern. Then, it becomes natural to extend to maps statistical studies on Dyck words (see [7,9,12,10,11,13] for instance). It is one of the main purpose of this paper.

A *Dyck word* on the alphabet $\{a, \bar{a}\}$ is a word generated by the context-free grammar $S \rightarrow \varepsilon \mid aS\bar{a}S$ where ε is the empty word. Let $\mathcal{D}_a$ be the set of all Dyck words on the alphabet $\{a, \bar{a}\}$, and let $\mathcal{D}_b$ be the set of all Dyck words on the alphabet $\{b, \bar{b}\}$. It is well known that Dyck words of semilength n are counted by Catalan numbers (A000108 in the on-line encyclopedia of integer sequences [14]), and that any non-empty Dyck word $w \in \mathcal{D}_a$ has a unique decomposition of the form $w = a\alpha\bar{a}\beta$ where α and β are two Dyck words in $\mathcal{D}_a$ (see [7]). Also, a word w on the alphabet $\{a, \bar{a}\}$ belongs to $\mathcal{D}_a$ if and only if the following two properties hold: (i) the number of letters a is equal to the number of letters $\bar{a}$ in w, and (ii) in any prefix of w, the number of letters a is greater than or equal to the number of letters $\bar{a}$.

We say that an occurrence of the letter a *matches* an occurrence of $\bar{a}$ located to its right in $w \in \mathcal{D}_a$, whenever the subword of w consisting of all letters between these two occurrences also belongs to $\mathcal{D}_a$. In this case, the pair $(a, \bar{a})$ is called a *matching* in w. For instance, if $w = a\bar{a}aa\bar{a}aa\bar{a}\bar{a}\bar{a}$, then the second occurrence of the letter a matches the last occurrence of $\bar{a}$ since $a\bar{a}aa\bar{a}\bar{a}$ is a Dyck word.

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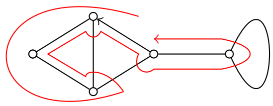
0012-365X/

a

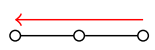
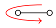
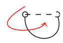
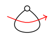
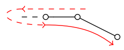


b

$aaabb\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}ab\bar{b}\bar{a}$

Fig. 1. A rooted planar map with 7 edges and its non-crossing shuffle representation.

Table 1
Correspondence between patterns in a shuffle and patterns in a rooted planar map.

Shuffle	a	aa	$a\bar{a}$	b	$b\bar{b}$	$\bar{a}a$	$a\bar{b}$
Map							Does not appear

A *shuffle* of two Dyck words $v \in \mathcal{D}_a$ and $w \in \mathcal{D}_b$ is a word s on the alphabet $\{a, \bar{a}, b, \bar{b}\}$ where s is constructed by interspersing the letters of v and w. Let $\mathcal{S}$ be the set of all shuffles of two Dyck words of $\mathcal{D}_a$ and $\mathcal{D}_b$. Shuffles of semilength n, $n \geq 0$, in $\mathcal{S}$ are enumerated by the sequence A005568 in [14]. The first values for $n \geq 0$ are 1, 2, 10, 70, 588, 5544, 56628. For instance, $s = aab\bar{a}\bar{a}\bar{b}ab\bar{b}\bar{a}$ is a shuffle of the two Dyck words $aa\bar{a}\bar{a}a\bar{a} \in \mathcal{D}_a$ and $b\bar{b}b\bar{b} \in \mathcal{D}_b$.

For any shuffle s, we denote by $w_a(s)$ (resp. $w_b(s)$) the Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$) obtained from s by deleting the letters b and $\bar{b}$ (resp. a and $\bar{a}$). In the following, we will extend the definition of $w_a(s)$ and $w_b(s)$ for any word in $\{a, \bar{a}, b, \bar{b}\}^*$. In particular, if s is a prefix of shuffles, $w_a(s)$ (resp. $w_b(s)$) becomes a prefix of a Dyck word in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$). For instance, if $s = aab\bar{a}\bar{a}\bar{b}ab\bar{b}\bar{a}$, then $w_a(s) = aa\bar{a}\bar{a}a\bar{a}$ and $w_b(s) = b\bar{b}b\bar{b}$; and if $s = aab\bar{a}$, it is a prefix of a shuffle, and $w_a(s) = aa\bar{a}$ and $w_b(s) = b$ are prefixes of Dyck words. Then, *a word s is a shuffle in $\mathcal{S}$ if and only if $w_a(s) \in \mathcal{D}_a$ and $w_b(s) \in \mathcal{D}_b$.*

A shuffle s of two Dyck words $v \in \mathcal{D}_a$ and $w \in \mathcal{D}_b$ will be called *crossing* whenever there exists a matching $(a, \bar{a})$ in v and a matching $(b, \bar{b})$ in w such that s can be written $s = \alpha b \beta a \gamma \bar{b} \delta \bar{a} \eta$ where α, β, γ, δ and η belong to $\{a, \bar{a}, b, \bar{b}\}^*$. Then the occurrence $ba\bar{b}\bar{a}$ will be called a *pair of crossing matchings*. Let $\mathcal{NCS} \subset \mathcal{S}$ be the subset of non-crossing shuffles in $\mathcal{S}$, *i.e.*, shuffles with no pair of crossing matchings. For instance, $a\bar{a}b\bar{b}a\bar{a}b\bar{b}a\bar{a}$ is in $\mathcal{NCS}$, while $a\bar{a}bb\bar{b}a\bar{b}\bar{a}$ is not in $\mathcal{NCS}$. Notice that non-crossing shuffles are called *canonical parenthesis-bracket systems* in [17]. The shuffles of semilength $n \geq 0$ in $\mathcal{NCS}$ are enumerated by the sequence A000168 in [14] whose first values for $n \geq 0$ are 1, 2, 9, 54, 378, 2916, 24057, 208494. They are in one-to-one correspondence with the rooted planar maps with n edges [5,6,8,17]. See Fig. 1 for an example of rooted planar map with its representation as a non-crossing shuffle in $\mathcal{NCS}$. This one-to-one correspondence is obtained by the following process. Starting with the root edge, we follow or cross all edges of the map by making its tour in counter-clockwise direction. Each edge in the map must be reached twice. If the final vertex of the encountered edge has not yet been considered, then we follow this edge and we write the letter a; otherwise, if the edge is reached for the first time, then we write the letter b and we cross it; in the other cases the edge is reached for the second time, and we write $\bar{a}$ (resp. $\bar{b}$) and we follow the edge (resp. we cross the edge) whenever the first assignment of the edge was the letter a (resp. b). Table 1 gives the correspondence between some patterns of length at most two in a shuffle with their meaning in terms of map. For instance, a pattern $b\bar{b}$ in a shuffle of $\mathcal{NCS}$ corresponds to an empty loop on a vertex of the corresponding rooted planar map. Also, the pattern $a\bar{b}$ cannot occur in the shuffle representation of a rooted planar map because it necessarily creates a pair of crossing matchings.

In a recent paper [3] (for equivalence classes of permutations see [2]), the authors introduced an equivalence relation on the set of Dyck words where *two Dyck words are equivalent whenever the positions of the occurrences of a given pattern are the same in both words.* They provided generating functions for the numbers of equivalence classes whenever the patterns are of length two: aa, $a\bar{a}$, $\bar{a}\bar{a}$ and $\bar{a}a$. See also [4] for a study of this equivalence relation on Motzkin words.

The motivation of this paper is to present a similar study for shuffles of Dyck words and for rooted planar maps considered with their representation by shuffles in $\mathcal{NCS}$. So, we define the equivalence relation on $\mathcal{S}$ and $\mathcal{NCS}$ for any pattern π of length at most two:

two shuffles with the same semilength are π-equivalent whenever they coincide on the positions of occurrences of the pattern π.

For instance, the shuffle $s = aabab\bar{b}\bar{a}b\bar{a}\bar{b}\bar{a}$ is $\bar{b}\bar{a}$-equivalent to $s' = a\bar{a}abb\bar{b}\bar{a}abb\bar{b}\bar{a}$ since the occurrences of $\bar{b}\bar{a}$ appear in positions 5 and 9 in s and s'.

In Section 2, we show that the problem of the enumeration of the π-equivalence classes in $\mathcal{NCS}$ is the same as in $\mathcal{S}$ whenever π is a pattern of length at most two that does not belong to the set $\{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. In Sections 3–6, we present enumerative results by providing generating functions for the number of π-equivalence classes when $\pi \notin \{\bar{b}a, b\bar{a}, a\bar{b}, \bar{a}b\}$. Using the one-to-one correspondence between rooted planar maps and non-crossing shuffles of Dyck words, this induces enumerative results for equivalence classes of rooted planar maps relatively to the positions of some patterns. See Table 2 for an overview of these results. Notice that the pattern $a\bar{b}$ does not appear in any shuffle of a rooted planar map, and that we did not succeed to obtain the number of π-equivalence classes in $\mathcal{S}$ and $\mathcal{NCS}$ for $\pi \in \{\bar{a}b, \bar{b}a, b\bar{a}\}$. So, we leave these cases as open problems.

2. Some preliminary results

In this section, we prove that for some specific patterns π the numbers of π-equivalence classes in $\mathcal{S}$ and $\mathcal{NCS}$ are the same.