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Forbidden subgraphs in the norm graph

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ABSTRACT

We show that the norm graph with n vertices about $\frac{1}{2}n^{2-1/t}$ edges, which contains no copy of the complete bipartite graph $K_{t,(t-1)!+1}$, does not contain a copy of $K_{t+1,(t-1)!-1}$.

1. Introduction

Let H be a fixed graph. The *Turán number* of H, denoted $ex(n, H)$, is the maximum number of edges a graph with n vertices can have, which contains no copy of H. The Erdős–Stone theorem from [5] gives an asymptotic formula for the Turán number of any non-bipartite graph, and this formula depends on the chromatic number of the graph H.

When H is a complete bipartite graph, determining the Turán number is related to the "Zarankiewicz problem" (see [3], Chap. VI, Sect. 2, and [6] for more details and references). In many cases even the question of determining the right order of magnitude for $ex(n, H)$ is not known.

Let $K_{t,s}$ denote the complete bipartite graph with t vertices in one class and s vertices in the other. The probabilistic lower bound for $K_{t,s}$

$$ex(n, K_{t,s}) \geqslant cn^{2-(s+t-2)/(st-1)}$$

is due to Erdős and Spencer [4]. Kővari, Sós and Turán [12] proved that for $s \geqslant t$

$$ex(n, K_{t,s}) \leqslant \tfrac{1}{2}(s-1)^{1/t}n^{2-1/t} + \tfrac{1}{2}(t-1)n. \quad (1.1)$$

The norm graph $\Gamma(t)$, which we will define in the next section, has n vertices and about $\frac{1}{2}n^{2-1/t}$ edges. In [1] (based on results from [11]) it was proven that the graph $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$, thus proving that for $s \geqslant (t-1)! + 1$,

$$ex(n, K_{t,s}) > cn^{2-1/t}$$

for some constant c.

In [2], it was shown that $\Gamma(4)$ contains no copy of $K_{5,5}$, which improves on the probabilistic lower bound of Erdős and Spencer [4] for $ex(n, K_{5,5})$. In this article, we will generalise this result and prove that $\Gamma(t)$ contains no copy of $K_{t+1,(t-1)!-1}$. For $t \geqslant 5$, this does not improve the probabilistic lower bound of Erdős and Spencer, but, as far as we are aware, it is however the deterministic construction of a graph with n vertices containing no $K_{t+1,(t-1)!-1}$ with the most edges.

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2. The norm graph

Suppose that $q = p^h$, where p is a prime, and denote by $\mathbb{F}_q$ the finite field with q elements. We will use the following properties of finite fields. For any $a, b \in \mathbb{F}_q$, $(a+b)^{p^i} = a^{p^i} + b^{p^i}$, for any $i \in \mathbb{N}$. For all $a \in \mathbb{F}_{q^i}$, $a^q = a$ if and only if $a \in \mathbb{F}_q$. Finally $N(a) = a^{1+q+\cdots+q^{k-1}} \in \mathbb{F}_q$, for all $a \in \mathbb{F}_{q^k}$, since $N(a)^q = N(a)$.

Let $\mathbb{F}$ denote an arbitrary field. We denote by $\mathbb{P}_n(\mathbb{F})$ the projective space arising from the $(n+1)$-dimensional vector space over $\mathbb{F}$. Throughout dim will refer to projective dimension. A point of $\mathbb{P}_n(\mathbb{F})$ (which is a one-dimensional subspace of the vector space) will often be written as $\langle u \rangle$, where u is a vector in the $(n+1)$-dimensional vector space over $\mathbb{F}$.

Let $\Gamma(t)$ be the graph with vertices $(a, \alpha) \in \mathbb{F}_{q^{t-1}} \times \mathbb{F}_q$, $\alpha \neq 0$, where (a, α) is joined to (a', α') if and only if $N(a+a') = \alpha\alpha'$. The graph $\Gamma(t)$ was constructed in [11], where it was shown to contain no copy of $K_{t,t!+1}$. In [1] Alon, Rónyai and Szabó proved that $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$. Our aim here is to show that it also contains no $K_{t+1,(t-1)!-1}$, generalising the same result for $t = 5$ presented in [2].

Let

$$V = \{(1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}) \mid a \in \mathbb{F}_{q^{t-1}}\} \subset \mathbb{P}_{2^{t-1}-1}(\mathbb{F}_{q^{t-1}}).$$

The set V is the affine part of an algebraic variety that is in turn a subvariety of the Segre variety

$$\Sigma = \underbrace{\mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1}_{t-1 \text{ times}},$$

where $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{F}_q)$. We briefly recall that a Segre variety is the image of the Segre embedding:

$$\sigma : (v_1, v_2, \ldots, v_k) \in \mathbb{P}_{n_1-1} \times \mathbb{P}_{n_2-1} \times \cdots \times \mathbb{P}_{n_k-1} \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k \in \mathbb{P}_{n_1 n_2 \cdots n_k - 1}$$

i.e. it is the set of points corresponding to the simple tensors. For the reader that is not familiar to tensor products we remark that, up to a suitable choice of coordinates, if $v_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n_i-1}^{i})$, then $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ is the vector of all possible products of type: $x_{j_1}^{(1)} x_{j_2}^{(2)} \cdots x_{j_k}^{(k)}$ (see [10] for an easy overview on Segre varieties over finite fields).

Then, the affine point $P_a = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}})$ has coordinates indexed by the subsets of $T := \{0, 1, \ldots, t-1\}$, where the S-coordinate is

$$\left(\prod_{i \in S} a^{q^i}\right),$$

for any non-empty subset S of T and

$$\prod_{i \in S} a^{q^i} = 1$$

when $S = \emptyset$ (see [13]).

Let $n = 2^{t-1} - 1$.

We order the coordinates of $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ so that if the ith coordinate corresponds to the subset S, then the $(n-i)$th coordinate corresponds to the subset $T \setminus S$.

Embed the $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ containing V as a hyperplane section of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by the equation $x_{n+1} = 0$.

Let b be the symmetric bilinear form on the $(n+2)$-dimensional vector space over $\mathbb{F}_{q^{t-1}}$ defined by

$$b(u, v) = \sum_{i=0}^{n} u_i v_{n-i} - u_{n+1} v_{n+1}.$$

Let $\perp$ be defined in the usual way, so that given a subspace Π of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$, $\Pi^{\perp}$ is the subspace of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by

$$\Pi^{\perp} = \{v \mid b(u, v) = 0, \text{ for all } u \in \Pi\}.$$

We wish to define the same graph $\Gamma(t)$, so that adjacency is given by the bilinear form. Let $P = (0, 0, 0, \ldots, 1)$. Let Γ' be a graph with vertex set the set of points on the lines joining the affine points of V to P obtained using only scalars in $\mathbb{F}_q$, distinct from P and not contained in the hyperplane $x_{n+1} = 0$. Join two vertices $\langle u \rangle$ and $\langle u' \rangle$ in Γ' if and only if $b(u, u') = 0$. It is a simple matter to verify that the graph Γ' is isomorphic to the graph $\Gamma(t)$ by the map $P_a + \alpha P \mapsto (a, \alpha)$ since

$$N(a+b) - \alpha\beta = \sum_{S \subseteq T} \prod_{i \in S,\, j \in T \setminus S} a^{q^i} b^{q^j} - \alpha\beta = b(u, v),$$

where

$$u = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}) + \alpha P,$$