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Some spectral properties of cographs

A. Mohammadian [a,*], V. Trevisan [b]

[a] *Departamento de Engenharia de Produção, Centro Federal de Educação Tecnológica Celso Suckow da Fonseca (CEFET/RJ), CEP 20271-110, Rio de Janeiro, RJ, Brasil*

[b] *Instituto de Matemática, UFRGS, CEP 91509-900, Porto Alegre, RS, Brasil*

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ABSTRACT

A graph containing no path of length three as an induced subgraph is called a cograph. In this article, we give a recursive definition of cographs in terms of the vertex duplication and co-duplication operations. We then establish that no cographs have eigenvalues in the interval $(-1, 0)$, generalizing the same known result for threshold graphs. As a consequence, we present combinatorial descriptions for the multiplicities of 0 and -1 as eigenvalues of cographs. This provides a short proof of a known result.

1. Introduction

Throughout this article, all graphs are assumed to be finite, undirected, and without loops or multiple edges. Let us first set some notation and terminology. Let G be a graph with the vertex set $\mathcal{V}(G)$ and the edge set $\mathcal{E}(G)$. The *adjacency matrix* of G, denoted by $\mathcal{A}(G)$, is a matrix whose entries are indexed by $\mathcal{V}(G) \times \mathcal{V}(G)$ and the (u, v)-entry is 1 if $\{u, v\} \in \mathcal{E}(G)$ and 0 otherwise. The eigenvalues of $\mathcal{A}(G)$ are considered as the eigenvalues of G. Since $\mathcal{A}(G)$ is a real symmetric matrix, all eigenvalues of G are real numbers. We denote the multiplicity of λ as an eigenvalue of G by $m(G; \lambda)$. For any vertex $v \in \mathcal{V}(G)$, let $\mathcal{N}_G(v) = \{x \mid \{v, x\} \in \mathcal{E}(G)\}$ and $\mathcal{N}_G[v] = \{v\} \cup \mathcal{N}_G(v)$. By *duplicating* (respectively, *co-duplicating*) a vertex $v \in \mathcal{V}(G)$, we mean adding to G a new vertex v' and joining v' to all vertices in $\mathcal{N}_G(v)$ (respectively, $\mathcal{N}_G[v]$). Two vertices $u, v \in \mathcal{V}(G)$ are called *twin* if $\mathcal{N}_G(u) = \mathcal{N}_G(v)$, and *co-twin* if $\mathcal{N}_G[u] = \mathcal{N}_G[v]$. A graph with no twin (respectively, co-twin) vertices is called *twin-free* (respectively, *co-twin-free*). We may define a relation $\mathcal{R}$ (respectively, $\mathcal{S}$) on $\mathcal{V}(G)$ with $u \,\mathcal{R}\, v$ (respectively, $u \,\mathcal{S}\, v$) if and only if u and v are twin (respectively, co-twin) vertices. We may associate a graph $\mathcal{R}(G)$ to G whose vertex set consists of all equivalence classes of $\mathcal{R}$ so that two vertices $[u]_{\mathcal{R}}$ and $[v]_{\mathcal{R}}$ of $\mathcal{R}(G)$ are adjacent if $\{u, v\} \in \mathcal{E}(G)$. Similarly, we may also associate a graph $\mathcal{S}(G)$ to G whose vertex set consists of all equivalence classes of $\mathcal{S}$ such that two vertices $[u]_{\mathcal{S}}$ and $[v]_{\mathcal{S}}$ of $\mathcal{S}(G)$ are adjacent if $\{u, v\} \in \mathcal{E}(G)$.

A graph without induced paths on four vertices is said to a *cograph*. Cographs have arisen in many disparate areas of mathematics and have been independently rediscovered by various researchers. For two graphs G_1 and G_2, the *disjoint union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph whose vertex set is a disjoint union of $\mathcal{V}(G_1)$ and $\mathcal{V}(G_2)$ and whose edge set is $\mathcal{E}(G_1) \cup \mathcal{E}(G_2)$, and, the *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph whose vertex set is a disjoint union of $\mathcal{V}(G_1)$ and $\mathcal{V}(G_2)$ and whose edge set is $\mathcal{E}(G_1) \cup \mathcal{E}(G_2) \cup \{\{v_1, v_2\} \mid v_1 \in \mathcal{V}(G_1) \text{ and } v_2 \in \mathcal{V}(G_2)\}$. It is well known that the class of cographs can be defined recursively as follows:

(i) The single-vertex graph is a cograph;
(ii) The disjoint union and the join of two cographs are cographs. (1)

* Corresponding author.
E-mail addresses: ali_m@ipm.ir (A. Mohammadian), trevisan@mat.ufrgs.br (V. Trevisan).

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For this result and several other combinatorial characterizations of cographs, we refer the reader to see [5] and the references given therein.

In this article, we demonstrate that the class of cographs can also be defined recursively as follows: (i) the single-vertex graph is a cograph; (ii) the graphs resulting by duplicating and co-duplicating a vertex of a cograph are cographs. Using this definition, we establish for any cograph G that $m(G; 0) = |\mathcal{V}(G)| - |\mathcal{V}(\mathcal{R}(G))|$ if G has no isolated vertex, and $m(G; -1) = |\mathcal{V}(G)| - |\mathcal{V}(\mathcal{S}(G))|$. These results are established in [2,9] by employing rather complicated methods. The result about the multiplicity of 0 has been conjectured by Torsten Sillke and it is firstly proven in [7] by an algorithmic proof. Another variation of the result about the multiplicity of 0 can be found in [4].

It has been established in [6] that threshold graphs, which form a special subclass of cographs, have no eigenvalue in the interval $(-1, 0)$. We will extend this result to cographs by a simple manner. Note that any interval of the real line contains some eigenvalues of graphs, since, more generally, any root of a real-rooted monic polynomial with integer coefficients occurs as an eigenvalue of some tree [8]. It is finally noteworthy that 0 and -1 as the eigenvalues of graphs play critical roles in spectral graph theory. For example, it is known from [1] that the multiplicity of each real number, except 0 and -1, in the eigenvalue spectrum of a connected graph G on $n \geqslant 5$ vertices is at most $n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$, while the upper bound for 0 and -1 are respectively $n - 2$ and $n - 1$.

2. The results

In our proof, we will apply the following theorem which is a consequence of Cauchy's interlacing theorem [3, Corollary 2.5.2]. We label the eigenvalues of a graph G on n vertices in non-increasing order and denote them by $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$.

Theorem 1. *Let G be a graph and $v \in \mathcal{V}(G)$. Let $G - v$ be the graph resulting from G by removing v and all edges incident to v. Then $\lambda_{i+1}(G) \leqslant \lambda_i(G - v) \leqslant \lambda_i(G)$ for each $i \in \{1, \ldots, |\mathcal{V}(G)| - 1\}$.*

The following theorem, which is the main tool of our proofs, provides a new recursive definition of the class of cographs.

Theorem 2. *Let $\mathscr{C}$ be the class of graphs defined recursively as follows:*

(i) *The single-vertex graph belongs to $\mathscr{C}$;*
(ii) *For any graph $G \in \mathscr{C}$ and any vertex $v \in \mathcal{V}(G)$, the graphs resulting from G by duplicating and co-duplicating v belong to $\mathscr{C}$.*

Then $\mathscr{C}$ is precisely the class of cographs.

Proof. First, we prove that each cograph is contained in $\mathscr{C}$. By the definition of $\mathscr{C}$, the single-vertex graph belongs to $\mathscr{C}$. In view of the definition of cographs presented in (1), it is enough to show that $\mathscr{C}$ is closed under the disjoint union and join operations. For simplicity, we denote the graph with the vertex set $\{v\}$ by $\overset{v}{\bullet}$. Let $G_1, G_2 \in \mathscr{C}$. Then, there are two sequences of duplication or co-duplication operations $\mathcal{A}_1, \ldots, \mathcal{A}_r$ and $\mathcal{B}_1, \ldots, \mathcal{B}_s$ such that $G_1 = \mathcal{A}_r \circ \cdots \circ \mathcal{A}_1(\overset{v_1}{\bullet})$ and $G_2 = \mathcal{B}_s \circ \cdots \circ \mathcal{B}_1(\overset{v_2}{\bullet})$. Hence, starting from $\overset{v_1}{\bullet}$ and then duplicating and co-duplicating the vertex v_1, we respectively get the graphs $\overset{v_1}{\bullet}\ \ \overset{v_2}{\bullet}$ and $\overset{v_1}{\bullet}\!\!-\!\!\!-\!\!\overset{v_2}{\bullet}$. Letting $\mathcal{T} = \mathcal{A}_r \circ \cdots \circ \mathcal{A}_1 \circ \mathcal{B}_s \circ \cdots \circ \mathcal{B}_1$, we have $G_1 \cup G_2 = \mathcal{T}(\overset{v_1}{\bullet}\ \ \overset{v_2}{\bullet})$ and $G_1 \vee G_2 = \mathcal{T}(\overset{v_1}{\bullet}\!\!-\!\!\!-\!\!\overset{v_2}{\bullet})$. This shows that $G_1 \cup G_2, G_1 \vee G_2 \in \mathscr{C}$ and it follows from (1) that $\mathscr{C}$ must contain all cographs.

Next, we prove that each element of $\mathscr{C}$ is a cograph. Suppose otherwise. Since the single-vertex graph is a cograph and in view of the definition of $\mathscr{C}$, there exists a cograph $H \in \mathscr{C}$ with a vertex $v \in \mathcal{V}(H)$ such that the graph G resulting from H by duplicating or co-duplicating v contains an induced path P with $|\mathcal{V}(P)| = 4$. Let $\{v'\} = \mathcal{V}(G) \setminus \mathcal{V}(H)$. Since H is cograph, we deduce that $v' \in \mathcal{V}(P)$. But then so does v, namely $v \in \mathcal{V}(P)$, since v is a duplicate or co-duplicate of v'. This is a contradiction, since P is an induced subgraph of G and P is both twin-free and co-twin-free. The proof of the theorem is now complete. □

Example 3. The cograph depicted in Fig. 1 is obtained by the following steps: (1) Consider the single-vertex graph $\overset{v_1}{\bullet}$; (2) Co-duplicate v_1 to get v_2; (3) Co-duplicate v_1 to get v_3; (4) Duplicate v_1 to get v_4; (5) Duplicate v_2 to get v_5; (6) Duplicate v_3 to get v_6.

Lemma 4. *Let G be a graph and $v \in \mathcal{V}(G)$. Let G' and G'' be the graphs resulting from G by duplicating and co-duplicating v, respectively. Then $m(G'; 0) = m(G; 0) + 1$ and $m(G''; -1) = m(G; -1) + 1$.*

Proof. Let $\{v'\} = \mathcal{V}(G') \setminus \mathcal{V}(G)$ and $\{v''\} = \mathcal{V}(G'') \setminus \mathcal{V}(G)$. Since the rows and columns of $\mathcal{A}(G')$ corresponding to v and v' are the same, $rank(\mathcal{A}(G')) = rank(\mathcal{A}(G))$. Using this fact that the multiplicity of λ as an eigenvalue of an $n \times n$ real symmetric matrix $\boldsymbol{M}$ is equal to $n - rank(\lambda \boldsymbol{I} - \boldsymbol{M})$, we conclude that $m(G'; 0) = m(G; 0) + 1$. Similarly, since the rows and columns of $\boldsymbol{I} + \mathcal{A}(G'')$ corresponding to v and v'' are the same, we find that $rank(\boldsymbol{I} + \mathcal{A}(G'')) = rank(\boldsymbol{I} + \mathcal{A}(G))$, implying $m(G''; -1) = m(G; -1) + 1$. □

The next theorem states a result on the distribution of the eigenvalues of cographs.