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Nesting *House*-designs

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ABSTRACT

A cycle of length 5 with a *chordal*, i.e. an edge joining two non-adjacent vertices of the cycle, is called a graph H_5 or also an *House-graph*. In this paper, the spectrum of House-systems nesting C_3-systems, C_4-systems, C_5-systems and together (C_3, C_4, C_5)-systems, of all admissible indices are completely determined, without exceptions.

1. Introduction

Let λK_v be the complete multigraph defined in a vertex-set X, $|X| = v$. Let G be a subgraph of λK_v. A *G-decomposition* of λK_v, of order v and index λ, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge-set of λK_v into subsets all of which yield subgraphs isomorphic to G. A *G-decomposition* of λK_v is also called a G-design, of order v and index λ. The classes of the partition $\mathcal{B}$ are said *blocks*. Important and interesting results about G-designs can be found in [5,10,12,13].

A cycle of length 5 with a *chordal*, i.e. an edge joining two not adjacent vertices of the cycle, will be called an *House-graph* and will be denoted by H_5. If $H_5 = (X, E)$, where $X = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}\, \{d, e\}, \{e, a\}, \{a, c\}\}$, we will denote such a graph by $[(a), b, (c), d, e]$.

Let $\Sigma = (X, \mathcal{B})$ be H_5-design of order v and index λ or an H_5-decomposition of the complete multigraph λK_v. When a graph $H_5 = [(a), b, (c), d, e]$ is a block of Σ with *multiplicity n*, it will be indicated by $[(a), b, (c), d, e]_{(n)}$. Similar concepts and symbolism are given in [3].

We say that Σ is:

– (1) C_3*-perfect* if the family of all the C_3-cycles having edges $\{a, b\}, \{b, c\}, \{a, c\}$ generates a C_3-design Σ' of order v and index μ;

– (2) C_4*-perfect*, if the family of all the C_4-cycles having edges $\{a, c\}, \{c, d\}, \{d, e\}, \{e, a\}$ generates a C_4-design Σ' of order v and index σ;

– (3) C_5*-perfect*, if the family of all the C_5-cycles having edges $\{a, b\}, \{b, c\}, \{c, d\}\, \{d, e\}, \{e, a\}$ generates a C_5-design Σ' of order v and index τ.

In the case (1), we say that Σ has indices (λ, μ). Similarly, in (2) its indices are (λ, σ) and in (3) (λ, τ). Similar definitions and symbolism is given in [1,2,6]. For *perfect* G-designs see also [8,11].

In every case, we say that Σ' is a system *nested* into Σ, and also that Σ is nesting Σ'.

We say that an H_5-design Σ, which is C_h*-perfect*, with indices (λ, μ), and C_k*-perfect* with indices (λ, σ), for $h, k = 3, 4, 5$, has indices (λ, μ, σ), and we will say that it is a (C_h, C_k)-perfect. Similarly, if Σ of index λ is C_3-perfect of index μ, C_4-perfect of index σ, and also C_5-perfect of index τ, we will say that Σ is (C_3, C_4, C_5)-perfect, of indices $(\lambda, \mu, \sigma, \tau)$.

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It is known [4] that:

Theorem 1.1. *An H_5-design of order v exists if and only if $v \equiv 0$, or 1, or 4, or 9 (mod 12), $v \geq 9$, with the possible exception of $v = 24$.*

Further, the spectrum of House-designs nesting C_4-systems, for every admissible indices, is determined in [3], where the authors proved that:

Theorem 1.2. *There exists a C_4-perfect H_5-design of order v and indices $(3, 2)$ if and only if $v \equiv 0$ or 1 (mod 4), $v \geq 5$.*

Theorem 1.3. *There exists a C_4-perfect H_5-design of order v and indices $(6, 4)$ if and only if $v \geq 5$.*

Theorem 1.4. *There exists a C_4-perfect H_5-design of order v, $v \geq 5$, and indices (λ, μ) such that $2\lambda = 3\mu$.*

In this paper we study the all possible nestings in House-systems, determining completely the spectrum in all the possible cases.

In what follows, to construct House-systems, we will use often the *difference-method*. This means that we fix as vertex-set $X = Z_v$ and, defined a *base-block* $[(a), b, (c), d, e]$, its *translates* will be all the blocks of type $[(a+i), b+i, (c+i), d+i, e+i]$, for every $i \in \mathbb{Z}_v$. For a given v, it will be $D(v) = \{|x - y| : x, y \in \mathbb{Z}_v, x \neq y\}$.

2. C_3-perfect H_5-designs of index (2, 1)

In this section, the spectrum of C_3-perfect H_5-designs of index (2, 1) is completely determined. We begin with the necessary conditions.

Theorem 2.1. *If $\Sigma = (X, \mathcal{B})$ is a C_3-perfect H_5-design of order v and indices (λ, μ), then:*
(1) $\lambda = 2\mu$;
(2) $|\mathcal{B}| = \mu \frac{v(v-1)}{6}$;
(3) *for* $\mu = 1$, *it is* $v \equiv 1, 3 \pmod 6$.

Proof. Let $\Sigma = (X, B)$ be a C_3-perfect H_5-design of order v and indices (λ, μ). If $\Sigma' = (X, B')$ is the C_3-system nested in Σ, necessarily: $\mathcal{B} = \mathcal{B}'$. Since
$|\mathcal{B}| = \lambda \frac{v(v-1)}{12}$, $|\mathcal{B}'| = \mu \frac{v(v-1)}{6}$,
(1) and (2) follow easily. For (3), consider that Σ' is a Steiner triple system of index 1. □

Now we determine the spectrum of C_3-perfect H_5-designs of index (2, 1), examining at first the case $v = 6h + 1$ and after the case $v = 6h + 3$.

Theorem 2.2. *For $\lambda = 2$, $\mu = 1$ and for every $v \equiv 1 \pmod 6$, $v \geq 7$, there exists a C_3-perfect H_5-design of order v and indices (2, 1).*

Proof. Let $v \equiv 1 \pmod 6$, $v \geq 7$. We can consider the following cases:
(1) $v \equiv 7 \pmod{18}$;
(2) $v \equiv 13, \pmod{18}$;
(3) $v \equiv 1 \pmod{18}$, $v \geq 19$.
(1) Let $v = 7$. It is: $D(7) = \{1, 2, 3\}$. Therefore, consider the block: $B = [(0), 3, (1), 4, 6]$. If $\mathcal{B}$ is the collection of all the translates of B, we can verify that $\Sigma = (\mathbb{Z}_7, \mathcal{B})$ is an H_5-design of order 7 and indices (2, 1). Further, since in B the differences $\{1, 2, 3\}$ cover, exactly one time, the edges of the C_3-cycle, it follows that Σ is C_3-perfect.
Let $v = 18k + 7$, for $k \geq 1$. Since $D = \{1, 2, \dots, 9k + 3\}$, it is possible to define the following $3k + 1$ base-blocks:
$B_{1,h} = [(0), 8k + 2h + 4, (3h + 1), 3k + 2, 3h + 3]$, for $h \in \{0, \dots, k - 1\}$;
$B_{2,h} = [(0), 6k + h + 3, (3h + 2), 9k + 2, 3k + 3h + 2]$, for $h \in \{0, \dots, k - 1\}$;
$B_{3,h} = [(0), 4k + 2h + 4, (3h + 3), 12k + 5, 6k + 3h + 4]$, for $h \in \{0, \dots, k - 1\}$;
$B_4 = [(0), 7k + 3, (3k + 1), 9k + 3, 18k + 6]$.
If $\mathcal{B}$ is the collection of all the translates of these base-blocks, we can verify that $\Sigma = (\mathbb{Z}_{18k+7}, \mathcal{B})$ is an H_5-design having indices (2, 1). Observe that, in the base-blocks, the differences $1, 2, \dots, 9k + 3$ cover, exactly one time, the edges of the C_3-cycles. Further, the number of base-blocks is $3k + 1$ and every of them generates $18k + 7$ translates. It follows that $|\mathcal{B}| = (3k + 1)(18k + 7)$ and Σ is C_3-perfect.
(2) Let $v = 13$. It is: $D = \{1, 2, \dots, 6\}$. Therefore, it is possible to define the two base-blocks: $B_1 = [(0), 4, (1), 7, 3]$, $B_2 = [(0), 7, (2), 4, 5]$. If $\mathcal{B}$ is the collection of all the translates of B_1 and B_2, we can verify that $\Sigma = (\mathbb{Z}_{13}, \mathcal{B})$ is an H_5-design having indices (2, 1). Further, since in B_1 and B_2 the differences $\{1, 3, 4\}$ and $\{2, 5, 6\}$ cover, exactly one time, respectively the edges of the two C_3-cycles, it follows that Σ is C_3-perfect.
Let $v = 18k + 13$, for $k \geq 1$. Since $D = \{1, 2, \dots, 9k + 6\}$, it is possible to define the following $3k + 2$ base-blocks:
$B_{1,h} = [(0), 4k + 2h + 4, (3h + 1), 3k + 2, 3h + 3]$, for $h \in \{0, \dots, k - 1\}$;
$B_{2,h} = [(0), 6k + h + 5, (3h + 2), 9k + 8, 3k + 3h + 5]$, for $h \in \{0, \dots, k - 1\}$;
$B_{3,h} = [(0), 8k + 2h + 8, (3h + 3), 12k + 8, 6k + 3h + 7]$, for $h \in \{0, \dots, k - 1\}$;