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# An optimal generalization of the Colorful Carathéodory theorem

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ABSTRACT

The Colorful Carathéodory theorem by Bárány (1982) states that given $d+1$ sets of points in $\mathbb{R}^d$, the convex hull of each containing the origin, there exists a simplex (called a 'rainbow simplex') with at most one point from each point set, which also contains the origin. Equivalently, either there is a hyperplane separating one of these $d+1$ sets of points from the origin, or there exists a rainbow simplex containing the origin. One of our results is the following extension of the Colorful Carathéodory theorem: given $\lfloor d/2 \rfloor + 1$ sets of points in $\mathbb{R}^d$ and a convex object $C$, then either one set can be separated from $C$ by a *constant* (depending only on $d$) number of hyperplanes, or there is a $\lfloor d/2 \rfloor$-dimensional rainbow simplex intersecting $C$.



## 1. Introduction

The goal of this paper is to study the behavior of *low-dimensional* simplices with respect to convex sets in $\mathbb{R}^d$. We examine a number of classical theorems in discrete geometry – Radon's theorem [15], Carathéodory's theorem [10], Colorful Carathéodory theorem [4] – and prove extensions that demonstrate the phenomenon of low-dimensional intersections.

**Three classical theorems.** One of the starting theorems in discrete geometry is the following result. For a set $P \subset \mathbb{R}^d$, let $\texttt{conv}(P)$ denote the convex hull of $P$.

**Theorem 1** (*Radon's Theorem*)**.** *Given any set $P$ of $d+2$ points in $\mathbb{R}^d$, one can partition $P$ into two sets $P_1$ and $P_2$ such that* $\texttt{conv}(P_1) \cap \texttt{conv}(P_2) \neq \emptyset$.

Note here that one of the two sets $P_1$ and $P_2$ can be large, e.g., $P_1$ can consist of $d+1$ points. So only the trivial bound $|P_1|, |P_2| \leq d+1$ holds. Therefore one cannot get a better upper bound on the dimension of the simplices $\texttt{conv}(P_1)$ or $\texttt{conv}(P_2)$.

We say a point $p$ can be *separated* from a convex set $C$ if there exists a hyperplane $h$ with $C$ and $p$ in the interior of the two different halfspaces defined by $h$.

**Theorem 2** (*Carathéodory's Theorem*)**.** *If a convex set $C$ intersects the convex hull of some point set $P$, then it also intersects a simplex spanned by $P$. Equivalently, either $P$ can be separated from $C$ with one hyperplane, or $C$ intersects the convex hull of some $(d+1)$ points of $P$.*

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Here we have stated the theorem in a slightly more general form; usually it is stated where $C$ is just a point.

A beautiful extension of Carathéodory's theorem was discovered by Imre Bárány [4]:

**Theorem 3** (*Colorful Carathéodory Theorem*). *Given $d+1$ sets of points $P_1, \dots, P_{d+1}$ in $\mathbb{R}^d$ and a convex set $C$ such that $C \cap \mathrm{conv}(P_i) \neq \emptyset$ for all $i = 1, \dots, d+1$, there exists a set $Q$ with $C \cap \mathrm{conv}(Q) \neq \emptyset$ and where $|Q \cap P_i| = 1$ for all $i$. Such a $Q$ is called a 'rainbow set'. Equivalently, either some $P_i$ can be separated from $C$ with one hyperplane, or $C$ intersects the convex hull of a rainbow set of $d+1$ points.*

This theorem is also commonly stated for the case where $C$ is a point, but the above slight generalization follows immediately from Bárány's proof technique [4]. Also, Carathéodory's theorem follows by applying the Colorful Carathéodory theorem to $d+1$ copies of the same point set.

**Our results.** The starting point of our work is the following well-known generalization of the Erdős–Szekeres theorem (see [17] and the references therein):

**Theorem 4** (*Generalized Erdős–Szekeres Theorem*). *Given positive integers $d, k, n$ such that $\lceil d/2 \rceil + 1 \leq k \leq d$, there exists an integer $n_0 = ES_d(n, k)$ such that any set of $n_0$ points in $\mathbb{R}^d$ contains a subset $P$ of size $n$ with the following property: the simplex spanned by every $(d+1) - k$ points of $P$ lies on the boundary of $\mathrm{conv}(P)$. This statement is optimal, in the sense that this is not true for $k < \lceil d/2 \rceil + 1$ for arbitrarily large point sets.*

The case $k = d$ simply corresponds to the Erdős–Szekeres theorem (that any large-enough set contains a lot of points in convex position). Of course the 'large-enough' size for the above theorem increases with decreasing $k$; but if one pays that price, one can get more properties. For example, for $d = 4, k = 3$, any large-enough set of points in $\mathbb{R}^4$ contains a large subset $Q$ where every edge spanned by points of $Q$ lies on $\mathrm{conv}(Q)$.

We now observe that this immediately carries over to an at-first nonobvious extension of Radon's theorem: if one is willing to increase the number of points, then a better upper-bound can be achieved on the sizes of the Radon partition:

**Theorem 1.1.** *Given integers $k$ and $d$ such that $\lfloor d/2 \rfloor + 1 \leq k \leq d$, any set $P$ of $ES_d(d+2, k)$ points in $\mathbb{R}^d$ contains two sets $P_1, P_2$ such that $\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) \neq \emptyset$ and additionally, $|P_1|, |P_2| \leq k$. Furthermore, this is optimal in the sense that the statement does not hold for $k \leq \lfloor d/2 \rfloor$.*

**Proof.** Apply Theorem 4 to $P$ to get a set of $d+2$ points $P'$. Apply Radon's theorem to $P'$ to get a partition $P_1, P_2 \subset P'$ whose convex hulls intersect. Now note that if $|P_1| > k$, then $|P_2| \leq (d+1) - k$. But then $\mathrm{conv}(P_2)$ lies on the convex hull of $P'$, and so cannot intersect $\mathrm{conv}(P_1)$, a contradiction.

Optimality is obvious as $|P| \geq d+2$ for such a partition to exist (for $P$ in general position), and so one set has to have at least $\lfloor d/2 \rfloor + 1$ points. □

Our first result is to show that a similar extension is possible for Carathéodory's theorem (Section 4):

**Theorem 1.2.** *Given a set $P$ of $n$ points in $\mathbb{R}^d$ and a convex object $C$, either $P$ can be separated from $C$ by $O(d^4 \log d)$ hyperplanes (i.e., each $p \in P$ is separated from $C$ by one of the hyperplanes), or $C$ intersects the convex hull of some $(\lfloor d/2 \rfloor + 1)$-sized subset of $P$.*

We show the above result by relating this problem to another well-known problem; in fact we prove that the bounds for these two problems are within a factor of $d$ of each other, a result of independent interest.

Unfortunately the above approach does not work for proving an extension of the Colorful Carathéodory theorem, for which we give a proof using a different technique (Section 5):

**Theorem 1.3.** *For any $d$, there exists a constant $N_d$ such that given $k = \lfloor d/2 \rfloor + 1$ sets of points $P_1, \dots, P_k$ in $\mathbb{R}^d$ and a convex object $C$, either one of the sets $P_i$ can be separated from $C$ by $N_d$ hyperplanes, or there is a rainbow set of size $k$ whose convex hull intersects $C$.*

**Remark 1.** Unlike the small polynomial bound in the extension of Carathéodory's theorem, the constant $N_d$ is exponential in $d$. We leave improving $N_d$ as an open problem.

**Remark 2.** The case where there are $d+1$ sets, and a set can be separated by one hyperplane is exactly the Colorful Carathéodory theorem.

**Remark 3.** Note also that, as before, Theorem 1.3 implies the corresponding extension for Carathéodory's theorem, although with much worse quantitative bound than given in Theorem 1.2.

We conclude with some open problems and future directions of research in Section 6.