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Diameter and maximum degree in Eulerian digraphs

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ABSTRACT

In this paper we consider upper bounds on the diameter of Eulerian digraphs containing a vertex of large degree. Define the maximum degree of a digraph to be the maximum of all in-degrees and out-degrees of its vertices. We show that the diameter of an Eulerian digraph of order n and maximum degree Δ is at most $n - \Delta + 3$, and this bound is sharp. We also show that the bound can be improved for Eulerian digraphs with no 2-cycle to $n - 2\Delta + O(\Delta^{2/3})$, and we exhibit an infinite family of such digraphs of diameter $n - 2\Delta + \Omega(\Delta^{1/2})$. We further show that for bipartite Eulerian digraphs with no 2-cycle the diameter is at most $n - 2\Delta + 3$, and that this bound is sharp.

1. Introduction

While distances in undirected graphs have been the subject of much research, far less is known about distances in directed graphs, which for many real world applications are a better model than undirected graphs. Many properties that hold for distances in undirected graphs do not hold for digraphs. For example the distance between vertices of a graph is a metric, but this is not true for digraphs in general.

Recently it has been shown that some bounds on the diameter of undirected graphs hold for all Eulerian digraphs, an important class of digraphs containing all graphs. A classical result on the maximum number of edges of a connected graph of given order and diameter by Ore [14] and Homenko and Ostroverhiĭ [11] has been shown in [6] to hold for all Eulerian digraphs. Soares [15] showed that the upper bound $\frac{3n}{\delta+1} + O(1)$ on the diameter of a connected graph of order n and minimum degree δ (see for example [1,9,10,13]) holds for all Eulerian digraphs. For further results on the diameter of Eulerian digraphs we refer the reader to [5,7,12], and for recent results relating the diameter of digraphs or graphs to other graph parameters see [2,4,8].

The aim of this paper is to give upper bounds on the diameter of Eulerian digraphs of given order and maximum degree. It is straightforward (see for example [3]) that for an undirected graph of order n and maximum degree Δ the diameter is bounded from above by $n+1-\Delta$. So a single vertex of sufficiently large degree in a graph reduces its diameter significantly. This is not the case for digraphs in general. For example the unique strong tournament of order n and diameter $n-1$ has vertices of out-degree $n-2$, in-degree $n-2$, and vertices whose out-degree and in-degree both equal approximately $\frac{n-1}{2}$, yet the diameter attains the maximum possible value $n-1$.

Define the maximum degree of a digraph to be the maximum of all in-degrees and all out-degrees of its vertices. In this paper we show that for all Eulerian digraphs of order n and maximum degree Δ the diameter is bounded from above by $n + 3 - \Delta$, which is sharp and only slightly weaker than the above-mentioned bound $n + 1 - \Delta$ for undirected graphs.

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We also show that for Eulerian oriented graphs, i.e., Eulerian digraphs with no 2-cycle, this bound can be improved to $n - 2\Delta + O(\Delta^{2/3})$, and that this is in some sense not far from best possible by constructing an infinite family of Eulerian oriented graph of diameter $n-2\Delta+\Omega(\Delta^{1/2})$. For bipartite Eulerian oriented graphs we improve the bound on the diameter further to $n - 2\Delta + 3$, which is sharp.

The notation and terminology we use is as follows. D always denotes a strongly connected, or strong for short, digraph with vertex set V and arc set E. The distance $d(u, v)$ between two vertices u and v of D is the length of a shortest (u, v)-path, and the diameter of D, denoted by $\mathrm{diam}(D)$, is the largest of all distances between vertices of D. For a given vertex v in a digraph and $i \in \mathbb{N}_0$ we denote the ith out-neighbourhood of v by $N_i(v)$, i.e., $N_i(v) = \{w \in V \mid d(v, w) = i\}$, and we let $n_i(v) = |N_i(v)|$. Furthermore we denote $\bigcup_{j=0}^{i} N_j(v)$ by $N_{\le i}(v)$, and $\bigcup_{j=i}^{d} N_j(v)$ by $N_{\ge i}(v)$, where d is the diameter of D. We sometimes drop the argument v if the vertex v is clear from the context and write N_i, $N_{\le i}$, and $N_{\ge i}$, respectively. The out-degree and in-degree of a vertex v, denoted by $\deg^+(v)$ and $\deg^-(v)$, respectively, is the number of out-neighbours and in-neighbours of v, and if the in-degree and out-degree of v coincide then we refer to it as the degree of v. The maximum degree Δ of a digraph D is the maximum of all in-degrees and out-degrees of the vertices of D. An arc joining a vertex u to a vertex v is denoted by $\overrightarrow{uv}$. For two sets A, B of vertices we write (A, B) for the set of all arcs $\overrightarrow{uv} \in E$ with $u \in A$ and $v \in B$.

A digraph D is Eulerian if it is strong and $\deg^+(v) = \deg^-(v)$ for every vertex of D. In our proofs we make frequent use of the fact that for every set $W \subset V$ with $\emptyset \ne W \ne V$ we have

$$|(W, V - W)| = |(V - W, W)|. \tag{1}$$

2. Results

Theorem 1. *Let D be an Eulerian digraph of order n and maximum degree Δ. Then*

$$\mathrm{diam}(D) \le n - \Delta + 3. \tag{2}$$

This bound is sharp for all $n \ge 10$ and $\Delta \ge 7$.

Proof. Let D be an Eulerian digraph of order n, maximum degree Δ and diameter d. Let v_0 and v_d be two vertices with $d(v_0, v_d) = d$, and $P : v_0, v_1, v_2, \ldots, v_d$ a shortest (v_0, v_d)-path. Let w be a vertex of D with $\deg^+(w) = \deg^-(w) = \Delta$. If $w = v_0$, then P contains exactly one out-neighbour of w since otherwise P would not be a shortest path. Hence, if $w = v_0$ we have $|V(P)| \le n - (\deg^+(w) - 1) = n - \Delta + 1$ and so $\mathrm{diam}(D) = |V(P)| - 1 \le n - \Delta$, and (2) holds. A similar argument shows that (2) holds if $w = v_d$. Hence we may assume from now on that $w \notin \{v_0, v_d\}$. Since we can choose any vertex in N_d to be the terminal vertex of a shortest path of length d, we may assume that $w \notin N_d$, and so $w \in N_r$ for some $r \in \{1, 2, \ldots, d-1\}$.

Let L and K be the set of in-neighbours and out-neighbours, respectively, of w that are on P. We may assume that

$$|L| \ge |K|. \tag{3}$$

Indeed, if $|K| > |L|$ then we consider not D but the digraph D' obtained from D by reversing all arcs, which is also Eulerian and has the same order, diameter, maximum degree, and the same path P, but with vertices in reverse order. For D' the corresponding sets K' and L' equal L and K, respectively, implying $|L'| > |K'|$, and thus (3).

Let $\ell = |L|$ and $v_{d_1}, v_{d_2}, \ldots, v_{d_\ell}$ the in-neighbours of w that are on P, where $d_1 < d_2 < \cdots < d_\ell$, and so $d_1 = r - 1$ and $d_3 > d_2 \ge r + 1$.

We partition the set of vertices not on P into two sets, A and B, where

$$A = N_{\le r+1} - V(P), \qquad B = N_{\ge r+2} - V(P).$$

Case 1: $w \in V(P)$.

In order to prove (2) it suffices to show that

$$|B| \ge |L| - 4, \tag{4}$$

since then, by $N^+(w) \subseteq A \cup K$ and (3), we have

$$|A| + |B| \ge |A| + |L| - 4 \ge |A| + |K| - 4 \ge \deg^+(w) - 4 = \Delta - 4,$$

and so

$$d = n - 1 - (|A| + |B|) \le n - \Delta + 3,$$

as desired. We may assume that $|L| \ge 5$, since otherwise (4) holds trivially.

Now let $j \in \mathbb{N}$ with $d_3 \le j \le d_{\ell-2} - 1$ and consider the set $(N_{\le j}, N_{\ge j+1})$. Since $w \in N_{\le j}$ and w has at least 3 out-neighbours in $N_{\ge j+1}$, viz $v_{d_{\ell-2}}, v_{d_{\ell-1}}$ and v_{d_ℓ}, we have $|(N_{\ge j+1}, N_{\le j})| \ge 3$. Hence $|(N_{\le j}, N_{\ge j+1})| \ge 3$ by (1). Since all arcs in $(N_{\le j}, N_{\ge j+1})$ join a vertex in N_j to a vertex in N_{j+1}, it follows that there are at most $n_j n_{j+1}$ such arcs, hence

$$n_j n_{j+1} \ge 3$$