Discrete Mathematics 339 (2016) 1382–1386

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

The difference and ratio of the fractional matching number and the matching number of graphs

Ilkyoo Choi [a], Jaehoon Kim [b], Suil O [c,*]

[a] *Department of Mathematical Sciences, KAIST, Daejeon, South Korea*

[b] *School of Mathematics, University of Birmingham, Edgbaston, Birmingham, United Kingdom*

[c] *Department of Mathematics, Simon Fraser University, Burnaby, BC, V5A 2R7, Canada*

ARTICLE INFO

Article history:
Received 8 April 2015
Accepted 5 December 2015
Available online 23 December 2015

Keywords:
Matching number
Fractional matching number

ABSTRACT

Given a graph G, the *matching number* of G, written $\alpha'(G)$, is the maximum size of a matching in G, and the *fractional matching number* of G, written $\alpha'_f(G)$, is the maximum size of a fractional matching of G. In this paper, we prove that if G is an n-vertex connected graph that is neither K_1 nor K_3, then $\alpha'_f(G) - \alpha'(G) \le \frac{n-2}{6}$ and $\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{3n}{2n+2}$. Both inequalities are sharp, and we characterize the infinite family of graphs where equalities hold.

1. Introduction

For undefined terms, see [5]. Throughout this paper, n will always denote the number of vertices of a given graph. A *matching* in a graph is a set of pairwise disjoint edges. A *perfect matching* in a graph G is a matching in which each vertex has an incident edge in the matching; its size must be $n/2$, where $n = |V(G)|$. A *fractional matching* of G is a function $\phi : E(G) \to [0, 1]$ such that for each vertex v, $\sum_{e\in\Gamma(v)} \phi(e) \le 1$, where $\Gamma(v)$ is the set of edges incident to v, and the *size of a fractional matching* ϕ is $\sum_{e\in E(G)} \phi(e)$. Given a graph G, the *matching number* of G, written $\alpha'(G)$, is the maximum size of a matching in G, and the *fractional matching number* of G, written $\alpha'_f(G)$, is the maximum size of a fractional matching of G.

Given a fractional matching ϕ, since $\sum_{e\in\Gamma(v)} \phi(e) \le 1$ for each vertex v, we have that $2\sum_{e\in E(G)} \phi(e) \le n$, which implies $\alpha'_f(G) \le n/2$. By viewing every matching as a fractional matching it follows that $\alpha'_f(G) \ge \alpha'(G)$ for every graph G, but equality need not hold. For example, the fractional matching number of a k-regular graph equals $n/2$ by setting weight $1/k$ on each edge, but the matching number of a k-regular graph can be much smaller than $n/2$. Thus it is a natural question to find the largest difference between $\alpha'_f(G)$ and $\alpha'(G)$ in a (connected) graph.

In Sections 3 and 4, we prove tight upper bounds on $\alpha'_f(G) - \alpha'(G)$ and $\frac{\alpha'_f(G)}{\alpha'(G)}$, respectively, for an n-vertex connected graph G, and we characterize the infinite family of graphs achieving equality for both results. As corollaries of both results, we have upper bounds on both $\alpha'_f(G) - \alpha'(G)$ and $\frac{\alpha'_f(G)}{\alpha'(G)}$ for an n-vertex graph G, and we characterize the graphs achieving equality for both bounds.

Our proofs use the famous Berge–Tutte Formula [1] for the matching number as well as its fractional analogue. We also use the fact that there is a fractional matching ϕ for which $\sum_{e\in E(G)} \phi(e) = \alpha'_f(G)$ such that $f(e) \in \{0, 1/2, 1\}$ for every edge e, and some refinements of the fact. We can prove both Theorems 6 and 8 with two different techniques, and for the sake of the readers we demonstrate each method in the proofs of Theorems 6 and 8.

* Corresponding author.
E-mail addresses: ilkyoo@kaist.ac.kr (I. Choi), kimJS@bham.ac.uk (J. Kim), suilo@gsu.edu (S. O).

http://dx.doi.org/10.1016/j.disc.2015.12.005

2. Tools

In this section, we introduce the tools we used to prove the main results. To prove Theorem 6, we use Theorems 1 and 2. For a graph H, let $o(H)$ denote the number of components of H with an odd number of vertices. Given a graph G and $S \subseteq V(G)$, define the *deficiency* $\mathrm{def}(S)$ by $\mathrm{def}(S) = o(G - S) - |S|$, and let $\mathrm{def}(G) = \max_{S\subseteq V(G)} \mathrm{def}(S)$. Theorem 1 is the famous Berge–Tutte formula, which is a general version of Tutte's 1-factor Theorem [4].

Theorem 1 *([1]). For any n-vertex graph G,* $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(G))$.

For the fractional analogue of the Berge–Tutte formula, let $i(H)$ denote the number of isolated vertices in H. Given a graph G and $S \subseteq V(G)$, let $\mathrm{def}_f(S) = i(G - S) - |S|$ and $\mathrm{def}_f(G) = \max_{S\subseteq V(G)} \mathrm{def}_f(S)$. Theorem 2 is the fractional version of the Berge–Tutte Formula. This is also the fractional analogue of Tutte's 1-Factor Theorem saying that G has a fractional perfect matching if and only if $i(G - S) \le |S|$ for all $S \subseteq V(G)$ (implicit in Pulleyblank [2]), where a fractional perfect matching is a fractional matching f such that $2\sum_{e\in E(G)} f(e) = n$.

Theorem 2 *([3] See Theorem 2.2.6). For any n-vertex graph G,* $\alpha'_f(G) = \frac{1}{2}(n - \mathrm{def}_f(G))$.

When we characterize the equalities in the bounds of Theorems 6 and 8, we need the following proposition. Recall that $G[S]$ is the graph induced by a subset of the vertex set S.

Proposition 3 *([3] See Proposition 2.2.2). The following are equivalent for a graph G.*
(a) *G has a fractional perfect matching.*
(b) *There is a partition $\{V_1, \ldots, V_n\}$ of the vertex set $V(G)$ such that, for each i, the graph $G[V_i]$ is either K_2 or Hamiltonian.*
(c) *There is a partition $\{V_1, \ldots, V_n\}$ of the vertex set $V(G)$ such that, for each i, the graph $G[V_i]$ is either K_2 or Hamiltonian graph on an odd number of vertices.*

Theorem 4 and Observation 5 are used to prove Theorem 8.

Theorem 4 *([3] See Theorem 2.1.5). For any graph G, there is a fractional matching f for which*

$$\sum_{e\in E(G)} f(e) = \alpha'_f(G)$$

such that $f(e) \in \{0, 1/2, 1\}$ for every edge e.

Given a fractional matching f, an *unweighted* vertex v is a vertex with $\sum_{e\in\Gamma(v)} f(e) = 0$, and a *full* vertex v is a vertex with $f(vw) = 1$ for some vertex w. Note that w is also a full vertex. An *i-edge* e is an edge with $f(e) = i$. Note that the existence of an 1-edge guarantees the existence of two full vertices. A vertex subset S of a graph G is *independent* if $E(G[S]) = \emptyset$, where $G[S]$ is the graph induced by S.

Observation 5. *Among all the fractional matchings of an n-vertex graph G satisfying the conditions of Theorem 4, let f be a fractional matching with the greatest number of edges e with $f(e) = 1$. Then we have the following:*
(a) *The graph induced by the $\frac{1}{2}$-edges is the union of odd cycles. Furthermore, if C and C′ are two disjoint cycles in the graph induced by $\frac{1}{2}$-edges, then there is no edge uu' such that $u \in V(C)$ and $u' \in V(C')$.*
(b) *The set S of the unweighted vertices is independent. Furthermore, every unweighted vertex is adjacent only to a full vertex.*
(c) $\alpha'(G) \ge w_1 + \sum_{i=1}^{\infty} ic_i$, $\alpha'_f(G) = w_1 + \sum_{i=1}^{\infty}(\frac{2i+1}{2})c_i$, *and* $n = w_0 + 2w_1 + \sum_{i=1}^{\infty}(2i + 1)c_i$, *where w_0, w_1, and c_i are the number of unweighted vertices, the number of 1-edges, and the number of odd cycles of length $2i + 1$ in the graph induced by $\frac{1}{2}$-edges in G, respectively.*

Proof. (a) The graph induced by the $\frac{1}{2}$-edges cannot have a vertex with degree at least 3 since $\sum_{e\in\Gamma(v)} f(e) \le 1$ for each vertex v. Thus the graph must be a disjoint union of paths or cycles. If the graph contains a path or an even cycle, then by replacing weight 1/2 on each edge on the path or the even cycle with weight 1 and 0 alternatively, we can have a fractional matching with the same fractional matching number and more edges with weight 1, which contradicts the choice of f. Thus the graph induced by the $\frac{1}{2}$-edges is the union of odd cycles. If there is an edge uv such that $u \in V(C)$ and $v \in V(C')$, where C and C' are two different odd cycles induced by some $\frac{1}{2}$-edges, then $f(uv) = 0$, since $\sum_{e\in\Gamma(x)} f(e) \le 1$ for each vertex x. By replacing weights 0 and 1/2 on the edge uv and the edges on C and C' with weight 1 on uv, and 0 and 1 on the edges in $E(C)$ and $E(C')$ alternatively, not violating the definition of a fractional matching, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction. Thus we have the desired result.
(b) If two unweighted vertices u and v are adjacent, then we can put a positive weight on the edge uv, which contradicts the choice of f. If there exists an unweighted vertex x, which is not incident to any full vertex, then x must be adjacent to a vertex y such that $f(yy_1) = 1/2$ and $f(yy_2) = 1/2$ for some vertices y_1 and y_2. By replacing the weights 0, 1/2, and 1/2 on xy, yy_1, and yy_2 with 1, 0, 0, respectively, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction.
(c) By the definitions of w_0, w_1, and c_i, we have the desired result. □