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Rainbow cycles in edge-colored graphs

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ABSTRACT

Let G be a graph of order n with an edge coloring c, and let $\delta^c(G)$ denote the minimum color degree of G, i.e., the largest integer such that each vertex of G is incident with at least $\delta^c(G)$ edges having pairwise distinct colors. A subgraph $F \subset G$ is rainbow if all edges of F have pairwise distinct colors. In this paper, we prove that (i) if G is triangle-free and $\delta^c(G) > \frac{n}{3} + 1$, then G contains a rainbow C_4, and (ii) if $\delta^c(G) > \frac{n}{2} + 2$, then G contains a rainbow cycle of length at least 4.

1. Introduction

We consider finite simple undirected graphs, and for notations and terminology not defined here we refer e.g. to [1]. By an *edge-colored graph* we mean a triple $G = (V(G), E(G), c)$, where $(V(G), E(G))$ is a (simple finite undirected) graph and $c : E(G) \to \mathbb{Z}^+$. The function c is called an *edge coloring* of G. If the edge coloring is clear from the context, we will simply speak of an edge-colored graph G. If $G = (V(G), E(G), c)$ and $H \subset G$ is a subgraph of G, then we automatically consider H to be edge-colored by the restriction of the function c to $E(H)$. An edge set $F \subset E(G)$ is called *rainbow* if no two distinct edges in F receive the same color, and a graph is called *rainbow* if its edge set is rainbow.

For an edge $e \in E(G)$, $c(e)$ denotes the color of e, and for a vertex $u \in V(G)$, $E(u)$ denotes the set of all edges incident to u. The cardinality of the set $c(E(u)) = \{c(e) : e \in E(u)\}$ is called the *color degree* of u and denoted by $d^c_G(u)$. The *minimum color degree* of G is denoted by $\delta^c(G)$ (or simply δ^c).

For $S_1, S_2 \subset V(G)$, $S_1 \cap S_2 = \emptyset$, we set $E(S_1, S_2) = \{xy \in E(G) : x \in S_1, y \in S_2\}$ and, in the special case when $S_1 = \{u\}$, we write $E(u, S)$ for $E(\{u\}, S)$. For a subgraph $F \subset G$ and a vertex $u \in V(G) \setminus V(F)$, we simply write $c(u, F)$ for $c(E(u, V(F)))$ and F^C for $G - F$.

For a subgraph $F \subset G$, we use $N_F(x)$ to denote the neighborhood of a vertex $x \in V(G)$ in F, i.e., the set of all vertices that are adjacent to x in F, and we write $d_F(x)$ for the degree of x in F. A path with endvertices $x, y \in V(G)$ is sometimes referred to as an (x, y)-path, and, for a path P and $u, v \in V(P)$, we use uPv to denote the subpath of P with endvertices u, v.

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The existence of rainbow substructures in edge-colored graphs has been widely studied in the literature. We mention here those of the known results that are related to our paper; for more information we refer the reader to the survey paper by Kano and Li [5].

It turns out that the problem of existence of rainbow cycles is closely related to the problem of existence of cycles in (uncolored) directed graphs. Let D be a directed graph with vertex set $V(D) = \{u_1, u_2, \ldots, u_p\}$ and arc set $A(D)$, and let G be the underlying (undirected) graph of D. Li [6] constructed an edge-coloring $c : E(G) \to V(D)$ of G by defining $c(u_iu_j) = u_j$ for each arc $u_iu_j \in A(D)$. It is easy to see that D has a directed cycle of length ℓ if and only if $(V(G), E(G), c)$ has a rainbow cycle of length ℓ. Hence the problem of the existence of rainbow cycles in edge-colored graphs is a generalization of the corresponding problem on directed cycles in directed graphs. Indeed, the problems on rainbow cycles seems to be more difficult than the directed problem. For instance, it is known that if the minimum out-degree $\delta^+(D)$ is at least one, then D has a directed cycle; however, for rainbow cycles, the corresponding problem to determine the minimum color degree which guarantees the existence of a rainbow cycle is not solved yet.

Li and Wang [7] constructed an edge-colored bipartite graph and an edge-colored complete graph, both having minimum color degree $\log_2 n$ and no rainbow cycles. Erdös and Gallai [4] showed that every graph G with $m(\geq n)$ edges has a cycle of length at least $\frac{2m}{n-1}$. For the number of colors $c(G) = |c(E(G))|$, Broersma et al. [2] pointed out that the Erdös and Gallai's theorem immediately implies that if $c(G) \geq n$, then G contains a rainbow cycle of length at least $\frac{2}{n-1}c(G)$. On the other hand, Li et al. [8] constructed an edge-coloring of the complete graph K_n as follows: let $V(K_n) = \{u_1, \ldots, u_n\}$ and let $c : E(K_n) \to V(K_n)$ be the edge-coloring defined by $c(u_iu_j) = u_j$ for all $i < j$. Then, obviously, K_n with this edge-coloring contains no rainbow cycle, and $c(K_n) = n - 1$. Thus, the lower bound of $c(G)$ obtained by Broersma et al. is best possible. Li et al. [8] showed that if $e(G) + c(G) \geq n(n+1)/2$, then G contains a rainbow triangle, and the above example implies that this lower bound of $e(G) + c(G)$ is also best possible.

For directed graphs, the following conjecture by Caccetta and Häggkvist [3] is well-known: a directed graph with $\delta^+ \geq d$ has a directed cycle of length at most $\lceil \frac{n}{d} \rceil$. For rainbow cycles, the following example can be considered as an extremal graph for the analogue of the Caccetta–Häggkvist conjecture. Let $V_1, \ldots, V_{d+1}$ be disjoint sets of vertices such that

$$\sum_{1\leq i\leq d+1} |V_i| = n \quad \text{and} \quad \lfloor n/(d+1) \rfloor \leq |V_i| \leq \lceil n/(d+1) \rceil \quad \text{for all } 1 \leq i \leq d+1,$$

and let $V_i = \{u^i_1, \ldots, u^i_{|V_i|}\}$ for each $1 \leq i \leq d+1$. Let G be the graph with the vertex set $\bigcup_{1\leq i\leq d+1} V_i$ and the edge set $\bigcup_{1\leq i\leq d-1}\{u^i_j u^{i+1}_k : u^i_j \in V_i, u^{i+1}_k \in V_{i+1}\}$. An edge-coloring $c : E(G) \to V(G)$ is defined by $c(u^i_j u^{i+1}_k) = u^{i+1}_k$ for all edges $u^i_j u^{i+1}_k \in E(G)$. Then obviously G contains no rainbow cycle of length at most d and $\delta^c = \lfloor \frac{n}{d+1} \rfloor + 1$. Thus, as in the case of directed graphs, we need a minimum color degree at least $\lfloor \frac{n}{d} \rfloor$ for the existence of a rainbow cycle of length at most d.

Our research is motivated by the following recent results. For short cycles, Broersma et al. [2] gave a neighborhood union-type condition by showing that if $|c(E(u) \cup E(v))| \geq n - 1$ for all pairs $u, v \in V(G)$, then G has a rainbow cycle of length at most four. A minimum degree condition for the existence of a rainbow triangle was given by Li [6].

Theorem A ([6]). *Let $(V(G), E(G), c)$ be an edge-colored graph of order $n \geq 3$. If*

$$\delta^c(G) > \frac{n}{2},$$

then G contains a rainbow triangle.

Li et al. [8] improved Theorem A as follows.

Theorem B ([8]). *Let $(V(G), E(G), c)$ be an edge-colored graph of order $n \geq 3$ satisfying one of the following conditions:*

(i) $\sum_{u\in V(G)} d^c(u) \geq \frac{n(n+1)}{2}$,
(ii) $\delta^c(G) \geq \frac{n}{2}$ *and* $G \notin \{K_{n/2,n/2}, K_4, K_4 - e\}$.

Then G contains a rainbow triangle.

Li [6] also gives the following condition for the existence of a rainbow 4-cycle in a balanced bipartite graph.

Theorem C ([6]). *Let $(V(G), E(G), c)$ be an edge-colored balanced bipartite graph of order $2n$. If*

$$\delta^c(G) > \frac{3n}{5} + 1,$$

then G contains a rainbow C_4.

The analogous question in edge-colored triangle-free (not necessarily bipartite) graphs was considered by Wang et al. [9].