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Note

The edge chromatic number of outer-1-planar graphs ☆

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ABSTRACT

A graph is outer-1-planar if each block has an embedding in the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this paper, we completely determine the edge chromatic number of outer 1-planar graphs.

1. Introduction

All graphs considered in this paper are simple and undirected. By $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, we denote the set of vertices, the set of edges, the maximum degree and the minimum degree of a graph G, respectively. In any figure of this paper, the degree of a solid or hollow vertex is exactly or at least the number of edges that are incident with it, respectively. Moreover, solid vertices are distinct but two hollow vertices may be identified unless stated otherwise.

A graph is *outer-1-planar* if each block has an embedding in the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. Outer-1-planar graphs were first introduced by Eggleton [3] who called them *outerplanar graphs with edge crossing number one*, and were also investigated under the notion of *pseudo-outerplanar graphs* by Zhang, Liu and Wu [11]. The notion of outer-1-planarity is a natural generalization of outer-planarity, and is also a combination of 1-planarity and outer-planarity. The definition of outer-1-planarity implies that outer-1-planar graphs are all planar.

It has been recently shown by Dehkordi and Eades [2] that every outer-1-planar graph has a right angle crossing drawing, and by Auer et al. [1] that the recognition of outer-1-planarity can process in linear time. On the other hand, the class of outer-1-planar graphs is used as a special graph class for verifying interesting conjectures on graph coloring. For instance, the list edge coloring conjecture and the list total coloring conjecture are verified for outer-1-planar graphs with maximum degree at least 5 [7,9], and the total coloring conjecture and the equitable Δ-coloring conjecture are confirmed for all outer-1-planar graphs [10,7].

An *edge k-coloring* of a graph G is an assignment $f : E(G) \to \{1, 2, \dots, k\}$ so that $f(e_1) \neq f(e_2)$ whenever e_1 and e_2 are two adjacent edges. The minimum integer k so that G has an edge k-coloring, denoted by $\chi'(G)$, is the *edge chromatic number* of G. The well-known Vizing's theorem states that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for every simple graph G. Therefore, to determine the edge chromatic number of a graph is interesting. However, this problem is NP-complete, and deciding whether a simple graph with maximum degree 3 has edge chromatic number 3 is still NP-complete [4]. As far as we know, the edge chromatic numbers of only few classes of graphs have been determined. For example, $\chi'(G) = \Delta(G)$ if G is a 1-planar graph with

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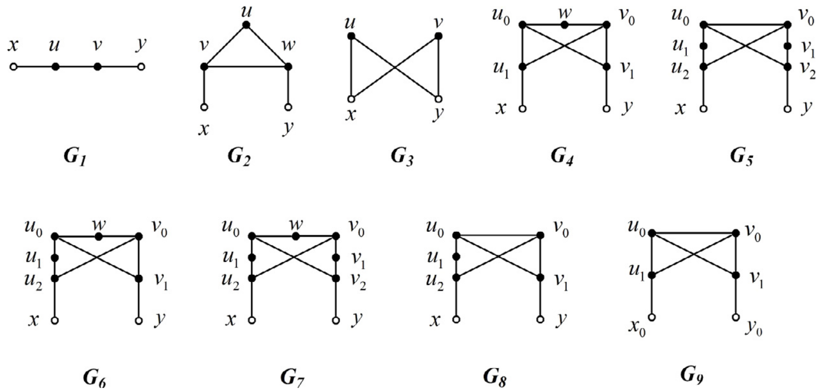

Fig. 1. Unavoidable structures in outer-1-planar graph with maximum degree at most 3.

maximum degree at least 10 [12], or a planar graph with maximum degree at least 7 [6], or a series–parallel graph (thus also an outer-planar graph) with maximum degree at least 3 [5].

The edge coloring of outer-1-planar graphs was first considered by Zhang, Liu and Wu [11]. They proved that the edge chromatic number of an outer-1-planar graph with maximum degree at least 4 is equal to the maximum degree, and announced that there are outer-1-planar graphs with maximum degree 3 and edge chromatic number 4 (the graph derived from K_4 by subdividing an edge is such an example). In this paper, we follow their work. First, we give structural results for outer-1-planar graphs with maximum degree at most 3. Next, we use the structural theorems to determine the edge chromatic number of all such outer-1-planar graphs.

2. Local structures of subcubic outer-1-planar graphs

Any outer-1-planar graph considered in this section is drawn in the plane so that its outer-1-planarity is satisfied and the number of crossings is as few as possible. We call such an outer-1-planar drawing an *outer-1-plane graph*. We follow the notation in [11].

Let G be 2-connected outer-1-plane graph. Denote by $v_1, v_2, \ldots, v_{|G|}$ the vertices of G that lie clockwise on the circle. Let $\mathcal{V}[v_i, v_j] := \{v_i, v_{i+1}, \ldots, v_j\}$ and let $\mathcal{V}(v_i, v_j) := \mathcal{V}[v_i, v_j] \setminus \{v_i, v_j\}$, where the subscripts are taken modulo $|G|$. Set $\mathcal{V}[v_i, v_i] := V(G)$. A vertex set $\mathcal{V}[v_i, v_j]$ with $i \neq j$ is *non-edge* if $j - i = 1 \pmod{|G|}$ and $v_iv_j \notin E(G)$, is *path* if $v_kv_{k+1} \in E(G)$ for all $i \leq k < j$, and is *subpath* if $j - i \neq 1 \pmod{|G|}$ and some edge in the form v_kv_{k+1} with $i \leq k < j$ is missing. An edge v_iv_j in G is *chord* if $j - i \neq 1 \pmod{|G|}$. By $\mathcal{C}[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$.

Lemma 2.1 ([11]). *Let v_i and v_j be vertices of a 2-connected outer-1-plane graph G. If there are no crossed chords in $\mathcal{C}[v_i, v_j]$ and no edges between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then $\mathcal{V}[v_i, v_j]$ is either non-edge or path.*

Subdividing an edge xy of a graph G means replacing xy with a path xzy with $d(z) = 2$. By K_4^+, K_4^{2a+}, K_4^{2b+} and K_4^{3+}, we denote the graph K_4 with one edge subdivided, two adjacent edges subdivided, two nonadjacent edges subdivided and a path of length 3 subdivided, respectively.

Theorem 2.2. *Every 2-connected outer-1-planar graph G with maximum degree at most 3 contains one of the configurations among $G_1, G_2, \ldots, G_8$ and G_9 as a subgraph, see Fig. 1, unless G is isomorphic to any of the graphs among K_4, K_4^+, K_4^{2a+}, K_4^{2b+} and K_4^{3+}. Moreover,*

(a) *if G contains G_2 with $x \neq y$ as a subgraph, then the graph derived from G by deleting u and identifying v with w is outer-1-planar;*

(b) *if G contains G_4 with $x \neq y$ as a subgraph, then the graph derived from G by deleting u_0, v_0, w and identifying u_1 with v_1 is outer-1-planar;*

(c) *if G contains G_8 with $x \neq y$ as a subgraph, then the graph derived from G by deleting u_0, u_1, v_0 and identifying u_2 with v_1 is outer-1-planar.*

Further, in the cyclic ordering of the vertices of G, the vertices of the configuration occur consecutively in the same order as drawn in the figure (up to symmetry in G_6 and G_8).

Proof. We prove this result by contradiction. If there are no crossings in G, then G is outer-planar and the result holds (cf. [8]). Therefore, there is at least one crossing in G.

Let v_iv_j and v_kv_l be two mutually crossed chords in G with $1 \leq i < k < j < l$. Without loss of generality, assume that there is no other pair of mutually crossed chords in $\mathcal{C}[v_i, v_l]$. By the outer-1-planarity of G and Lemma 2.1, each of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is either non-edge or path.