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On the type(s) of minimum size subspace partitions

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ABSTRACT

Let $V = V(kt+r, q)$ be a vector space of dimension $kt+r$ over the finite field with q elements. Let $\sigma_q(kt+r, t)$ denote the minimum size of a subspace partition $\mathcal{P}$ of V in which t is the largest dimension of a subspace. We denote by n_{d_i} the number of subspaces of dimension d_i that occur in $\mathcal{P}$ and we say $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$ is the *type* of $\mathcal{P}$. In this paper, we show that a partition of minimum size has a unique partition type if $t+r$ is an even integer. We also consider the case when $t+r$ is an odd integer, but only give partial results since this case is indeed more intricate.

1. Introduction

Let $V = V(n, q)$ denote a vector space of dimension $n = kt + r$ over the finite field with q elements. A *subspace partition* $\mathcal{P}$ of V, also known as a *vector space partition*, is a collection of non-zero subspaces of V such that each *point*, that is, 1-dimensional subspace, of V is in exactly one subspace of $\mathcal{P}$. We denote by n_{d_i} the number of subspaces of dimension d_i that occur in $\mathcal{P}$ and we say $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$ is the *type* of $\mathcal{P}$, where $d_1 < \cdots < d_m$ and $n_i > 0$ for $1 \le i \le m$. The *size* of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$. Let $\sigma_q(n, t)$ denote the *minimum size* of a subspace partition of V in which the largest subspace has dimension t. We call such a subspace partition a *t-minimal partition* of V.

Generalizing a theorem in [8], the following theorem was proved by the authors of the present paper in [4].

Theorem 1. *Let $n, k, t,$ and r be integers such that $1 \le r < t$, $k \ge 1$, and $n = kt + r$. Then*

$$\sigma_q(n, t) = q^t + 1 \quad \text{for } n < 2t,$$

and

$$\sigma_q(n, t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^{\lceil \frac{t+r}{2} \rceil} + 1 \quad \text{for } n \ge 2t.$$

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The question studied here is whether or not a subspace partition of minimum size, that is, attaining the lower bound given in Theorem 1, is of a type which just depends on n and t. We found that this is true when $t+r$ is an even integer. The case when $t+r$ is odd turned out to be more intricate and we obtain just partial answers in this case.

Let $\ell = q^r \sum_{i=0}^{k-2} q^{it}$. Our main results are thus the following theorems:

Theorem 2. *Let n, k, t, and r be integers such that $1 \le r < t$, $k \ge 2$, $t+r = 2s$ for some integer s, and $n = kt+r$. Let $\mathcal{P}$ be a t-minimal subspace partition of $V(n,q)$. Then $\mathcal{P}$ has type $[s^{n_s}, t^{n_t}]$, where*

$$n_s = q^s + 1, \quad \text{and} \quad n_t = \ell q^t.$$

Theorem 3. *Let n, k, t, and r be integers such that $r = t-1$, $k \ge 2$, and $n = kt+r$. Let $\mathcal{P}$ be a t-minimal subspace partition of $V(n,q)$. If $\mathcal{P}$ consists of spaces of two different dimensions, then $\mathcal{P}$ has type $[(t-1)^{n_{t-1}}, t^{n_t}]$, where*

$$n_{t-1} = q^t, \quad \text{and} \quad n_t = \ell q^t + 1.$$

When $r < t-1$ we obtain the following result:

Theorem 4. *Let n, k, t, and r be integers such that $1 \le r < t-1$, $k \ge 2$, $n = kt+r$, and $t+r = 2s-1$ for some integer s. Let $\mathcal{P}$ be a t-minimal subspace partition of $V(n,q)$. If the number of subspaces of dimension t is $n_t = \ell q^t$, then $\mathcal{P}$ has type $[(s-1)^{n_{s-1}}, s^1, t^{n_t}]$, where*

$$n_{s-1} = q^s, \quad \text{and} \quad n_t = \ell q^t.$$

It must be remarked that subspace partitions of types as indicated in the three previous theorems indeed exist and are well known, see Section 2.1 for a construction of them.

When $t+r$ is odd and $n_t < \ell q^t$, we were able to derive only two new non-trivial necessary conditions:

Theorem 5. *Let n, k, t, and r be integers such that $1 \le r < t-1$, $k \ge 2$, and $t+r = 2s-1$ for some integer s. Let $\mathcal{P}$ be a t-minimal subspace partition of $V(n,q)$.*

(1) *If $\mathcal{P}$ has type $[a^{n_a}, t^{n_t}]$, then $a = s = t-1$, i.e., $r = t-3$,*

$$n_a = q^{t-1} + q^{t-2} + 1, \quad \text{and} \quad n_t = \ell q^t - q^{t-2}.$$

(2) *If $\mathcal{P}$ has type $[a^{n_a}, b^{n_b}, t^{n_t}]$, then $a = s-1$, $b = s$,*

$$n_a = q^s - \delta q \frac{q^{t-s}-1}{q-1}, \qquad n_b = \delta \frac{q^{t-s+1}-1}{q-1} + 1, \quad \text{and} \quad n_t = \ell q^t - \delta,$$

for some integer δ such that

$$0 \le \delta \le \frac{(q^{s-2}-1)(q-1)}{q^{t-s}-1}.$$

In search for a reasonable conjecture when $t+r$ is odd, and also for the sake of exploring new methods, we first did a computer search in the particular case when $q = 2$, $t = 5$, $k = 2$ and $r = 2$, by using the Simplex Algorithm on the known necessary linear constraints for existence of subspace partitions. These linear constraints were found by Lehmann and Heden in [7]. This search showed that in these cases the type of a minimum size subspace partition is unique. This experience led us to the following conjecture:

Conjecture 1. *Let n, k, t, and r be integers such that $1 \le r < t-1$, $k \ge 2$, and $t+r = 2s-1$ for some integer s. Every t-minimal subspace partition $\mathcal{P}$ of $V(n,q)$ is of type*

$$[(t-1)^{q^{t-1}+q^{t-2}+1}, t^{\ell q^t - q^{t-2}}] \quad \text{or} \quad [(s-1)^{q^s}, s^1, t^{\ell q^t}].$$

As we found the linear programming approach fruitful, we tried to use it in the more general situation, when $t > 3$, $r = t-3$, and q is any prime power. Unfortunately, this led to rather complicated expressions that were tedious to evaluate. However, by using that approach we were able to prove Conjecture 1 for the special case when $k = 2$. The details of that proof might be published elsewhere.

2. Some preliminary results

2.1. The ideal partition

The following lemma due to Herzog and Schönheim [6] and independently Beutelspacher [1] and Bu [2], ensures the existence of a t-minimal subspace partition $\mathcal{P}_0$ of V for any $t \ge 2$.