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Induced hourglass and the equivalence between hamiltonicity and supereulerianity in claw-free graphs

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ABSTRACT

A graph H has the *hourglass property* if in every induced hourglass S (the unique simple graph with the degree sequence $(4, 2, 2, 2, 2)$), there are two non-adjacent vertices which have a common neighbor in $H - V(S)$. Let G be a claw-free simple graph and k a positive integer. In this paper, we prove that if either G is hourglass-free or G has the hourglass property and $\delta(G) \geq 4$, then G has a 2-factor with at most k components if and only if it has an even factor with at most k components. We provide some of its applications: combining the result (the case when $k = 1$) with Jaeger (1979) and Chen et al. (2006), we obtain that every 4-edge-connected claw-free graph with the hourglass property is hamiltonian and that every essentially 4-edge-connected claw-free hourglass-free graph of minimum degree at least three is hamiltonian, thereby generalizing the main result in Kaiser et al. (2005) and the result in Broersma et al. (2001) respectively in which the conditions on the vertex-connectivity are replaced by the condition of (essential) 4-edge-connectivity. Combining our result with Catlin and Lai (1990), Lai et al. (2010) and Paulraja (1987), we also obtain several other results on the existence of a hamiltonian cycle in claw-free graphs in this paper.

1. Introduction

In the hamiltonian theory, there exists a long-standing famous conjecture posed by Matthews and Sumner in [12] that every 4-connected claw-free graph is hamiltonian. Here a graph is called *claw-free* if it has no induced subgraph isomorphic to the claw $K_{1,3}$. The following result makes an attempt to this conjecture by considering a subset of graphs called *hourglass-free graphs* (the results in [11] show what role the property plays in hamiltonian-connectedness of a graph), *i.e.*, having no induced subgraph isomorphic to the *hourglass* (the unique simple graph with the degree sequence $(4, 2, 2, 2, 2)$).

Theorem 1 (*Broersma, Kriesell and Ryjáček, [2]*). *Every 4-connected claw-free hourglass-free graph is hamiltonian.*

Kaiser et al. extended the above result by considering graphs with the hourglass property, *i.e.*, the graph H such that in every induced hourglass S, there are two non-adjacent vertices which have a common neighbor in $H - V(S)$.

Theorem 2 (*Kaiser, Li, Ryjáček and Xiong, [9]*). *Every 4-connected claw-free graph with the hourglass property is hamiltonian.*

A natural question is whether the conditions on connectivity in the above two theorems can be relaxed. We give a positive answer to this question by proving the following result (the case when $k = 1$), where a spanning subgraph H of a graph G is

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called 2-*factor* if every vertex of H has degree 2; while it is called *an even factor* if every vertex of H has positive even degree. In particular, in Section 4, we show that the condition "4-connected" can be replaced by "essentially 4-edge-connected" (a graph G is called *essentially* 4-*edge-connected* if after deleting at most three edges of G the resulting graph has at most one component with at least one edge) and "4-edge-connected" in Theorems 1 and 2 respectively.

Theorem 3. *Suppose G is a simple claw-free graph and k is a positive integer. If either*

- *G is hourglass-free or,*
- *G has the hourglass property and is of minimum degree at least 4,*

then G has a 2-factor with at most k components, if and only if it has an even factor with at most k components.

We determine all the connected graphs H of order at least three guaranteeing the truth of the statement that a claw-free H-free graph G has a 2-factor with at most k components if and only if it has an even factor with at most k components. In particular, we show that the graphs H satisfying the above condition should be an induced connected subgraph of an induced hourglass. To see this, we consider the graph G_0 obtained from a triangle and a complete graph of order at least three by identifying exactly one vertex of the triangle and exactly one vertex of the other complete graph. Note that G_0 is not hamiltonian but it is a supereulerian claw-free graph. Therefore G_0 is not H-free, or H is one of its induced subgraphs. We also need consider the graph G_{00} obtained from a path $u_1u_2\dots u_t$ and $t-1$ additional vertices $v_1, v_2, \dots, v_{t-1}$ by joining each v_i with u_i and u_{i+1}. Note that G_{00} is nonhamiltonian but is supereulerian claw-free graph. Therefore G_{00} is not H-free, or H is one induced subgraph of G_{00}. Since G_0 has only one induced hourglass and H is one induced subgraph of G_{00}, the only possibility is that H should be one induced connected subgraph of an induced hourglass.

2. Preliminaries

2.1. Notation and terminology

For graph-theoretic notation not explained in this paper, we refer the reader to [1]. We consider only simple graph in this paper. Let $G=(V,E)$ be a graph with vertex set V and edge set E. For a subgraph H of G, we define $\bar{E}_G(H)$ to be the set of edges in G that are incident to some vertex in H. A subgraph H of G is called a *dominating subgraph* of G if $E(G)=\bar{E}_G(H)$. The *distance* of two vertices x, y in G, denoted by $d_G(x,y)$, is the length of a shortest path between x and y. Given two subgraphs G_1 and G_2, we define the distance $d_G(G_1,G_2)$ between G_1 and G_2 by $d_G(G_1,G_2)=\min\{d_G(x_1,x_2): x_1\in V(G_1), x_2\in V(G_2)\}$. For a vertex u of G, let $N_G(u)$ and $d_G(u)$ denote the set of neighbors of u and the degree of u in G, respectively. We use $\delta(G)$ to denote the minimum degree of G. An *even graph* is a graph in which every vertex has an even degree. A hamiltonian cycle is a 2-factor with exactly one component. A connected even subgraph is called *eulerian*. A graph is called *supereulerian* if it has a spanning eulerian subgraph; it has an even factor with exactly one component. A subgraph is called *trivial* if it has only one vertex; *nontrivial* otherwise.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.

2.2. Line graph closure operation and the stability of properties

Ryjáček [14] invented the concept of the *line graph closure* on claw-free graphs. Let G be a claw-free graph. If x is a locally connected vertex of G, then the *local completion at x* is the operation of adding all possible edges between vertices in $N_G(x)$. The resulting graph, denoted by G'_x, is easily shown to be claw-free again. Iterating local completions, we finally arrive at a graph in which all locally connected vertices have complete neighborhoods. This graph is uniquely determined (which is not quite obvious); it is called the *closure* of G and denoted by $cl(G)$.

Theorem 4 (*Ryjáček, [14]; Ryjáček, Saito and Schelp, [15]*)**.** *For a claw-free graph G,*

- *the (well-defined) closure $cl(G)$ of G is the line graph of some triangle-free graph,*
- *G is hamiltonian if and only if $cl(G)$ is hamiltonian,*
- *G has a 2-factor with at most k components if and only if $cl(G)$ has a 2-factor with at most k components.*

The following stable properties of the closure operation will enable us to restrict our attention to the line graphs.

Lemma 5 (*Brousek, Ryjáček and Schiermeyer, [3]*)**.** *Let G be a claw-free hourglass-free graph. Then $cl(G)$ is also hourglass-free.*

Lemma 6 (*Kaiser, Li, Ryjáček and Xiong, [9]*)**.** *Let G be a claw-free graph having the hourglass property. Then $cl(G)$ has also the hourglass property.*

Lemma 7 (*Xiong, [16]*)**.** *Let G be a claw-free graph. Then G has an even factor with at most k components if and only if $cl(G)$ has also an even factor with at most k components.*