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Counting paths in corridors using circular Pascal arrays

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ABSTRACT

A circular Pascal array is a periodization of the familiar Pascal's triangle. Using simple operators defined on periodic sequences, we find a direct relationship between the ranges of the circular Pascal arrays and numbers of certain lattice paths within corridors, which are related to Dyck paths. This link provides new, short proofs of some nontrivial formulas found in the lattice-path literature.

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1. Circular Pascal arrays and corridor paths

1.1. Circular Pascal arrays

We begin by defining the *circular Pascal arrays* (one for each integer $d \geq 2$) and explore some of their amazing properties. By the *Pascal array*, we mean something a little more general than the familiar Pascal's triangle. In what follows, we interpret the binomial coefficient $\binom{n}{k}$ as the coefficient of x^k in the expansion,

$$(1+x)^n = \sum_{k \in \mathbb{Z}} \binom{n}{k} x^k,$$

where we understand $\binom{n}{k} = 0$ if $k < 0$ or $k > n$. We also use the convention that $\mathbb{N}$ denotes the set of *non-negative integers*, $\{0, 1, 2, 3, \ldots\}$.

Definition 1. The *Pascal array* is the array whose row n, column k entry is equal to $\binom{n}{k}$ where $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

$n \setminus k$	$\cdots$	-1	0	1	2	3	4	5	$\cdots$
0	$\cdots$	0	1	0	0	0	0	0	$\cdots$
1	$\cdots$	0	1	1	0	0	0	0	$\cdots$
2	$\cdots$	0	1	2	1	0	0	0	$\cdots$
3	$\cdots$	0	1	3	3	1	0	0	$\cdots$
4	$\cdots$	0	1	4	6	4	1	0	$\cdots$
5	$\cdots$	0	1	5	10	10	5	1	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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Recall that for any $n > 0$, entry (n, k) of the array can be found by the familiar formula,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1}$$

Definition 2. Fix an integer $d \geq 2$. The *circular Pascal array of order d* is the array whose row n, column k entry, $\sigma^{(d)}_{n,k}$ (or just $\sigma_{n,k}$ when the context is clear), is the (finite) sum:

$$\sigma_{n,k} = \sigma^{(d)}_{n,k} = \sum_{j\in\mathbb{Z}} \binom{n}{k+dj}, \tag{2}$$

where $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

In what follows, we will be interested in *periodic* sequences and arrays of numbers. To be precise, we say a sequence $(a_k)_{k\in\mathbb{Z}}$ is *periodic*, of period d, if $a_{k+dm} = a_k$ for every $m \in \mathbb{Z}$. Clearly, for fixed $n \geq 0$, the sequence $(\sigma^{(d)}_{n,k})_{k\in\mathbb{Z}}$ as defined in (2) is periodic of period d. Furthermore, it is easy to see that the entries of the circular Pascal array satisfy (1), in the sense that for $n > 0$,

$$\sigma_{n,k} = \sigma_{n-1,k-1} + \sigma_{n-1,k}. \tag{3}$$

For $d = 5$, our collection of periodic sequences is indicated below.

$n \setminus k$	···	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	···
0	···	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	···
1	···	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0	···
2	···	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0	···
3	···	1	3	3	1	0	1	3	3	1	0	1	3	3	1	0	···
4	···	1	4	6	4	1	1	4	6	4	1	1	4	6	4	1	···
5	···	2	5	10	10	5	2	5	10	10	5	2	5	10	10	5	···
6	···	7	7	15	20	15	7	7	15	20	15	7	7	15	20	15	···
7	···	22	14	22	35	35	22	14	22	35	35	22	14	22	35	35	···
8	···	57	36	36	57	70	57	36	36	57	70	57	36	36	57	70	···
9	···	127	93	72	93	127	127	93	72	93	127	127	93	72	93	127	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Henceforth, we will represent a circular Pascal array by showing only columns $0, 1, \ldots, d-1$.

While studying a related problem called the Sharing Problem, Charles Kicey, Katheryn Klimko, and Glen Whitehead [2] noticed that the circular Pascal array has surprising connections to well-known sequences for small values of d. Consider the case $d = 2$ (shown below, along with $d = 3$ and $d = 4$). Starting in row $n = 1$, $\sigma_{n,k} = 2^{n-1}$ for $k = 0, 1$. Of course, this reflects a well-known property of Pascal's triangle: $\sum_{j\in\mathbb{Z}} \binom{n}{2j} = \sum_{j\in\mathbb{Z}} \binom{n}{2j+1}$, if $n \geq 1$. Another way to state this result is to say that the *range* (difference between maximum and minimum values) of the nth row is 0 for $n \geq 1$ in the circular Pascal array of order 2. The cases $d = 3, 4$ are interesting as well – for $d = 3$, the ranges are constantly 1, while for $d = 4$, the range of row n is $2^{\lfloor n/2 \rfloor}$ – but the most surprising case is, perhaps, $d = 5$. These ranges form the Fibonacci sequence, as was proved in [2]. Now Fibonacci numbers are no strangers to Pascal's triangle; indeed the sequence of diagonal sums, $f_n = \sum_{j\in\mathbb{Z}} \binom{n-j}{j}$, is easily shown to be the Fibonacci sequence. However, this new manifestation of the Fibonacci numbers in the circular Pascal array of order 5 was quite unexpected.

$d = 2$			
$n \setminus k$	0	1	Range
0	1	0	1
1	1	1	0
2	2	2	0
3	4	4	0
4	8	8	0
5	16	16	0
6	32	32	0
⋮	⋮	⋮	⋮

$d = 3$				
$n \setminus k$	0	1	2	Range
0	1	0	0	1
1	1	1	0	1
2	1	2	1	1
3	2	3	3	1
4	5	5	6	1
5	11	10	11	1
6	22	21	21	1
⋮	⋮	⋮	⋮	⋮

$d = 4$					
$n \setminus k$	0	1	2	3	Range
0	1	0	0	0	1
1	1	1	0	0	1
2	1	2	1	0	2
3	1	3	3	1	2
4	2	4	6	4	4
5	6	6	10	10	4
6	16	12	16	20	8
⋮	⋮	⋮	⋮	⋮	⋮