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New lower bounds on independence number in triangle-free graphs in terms of order, maximum degree and girth

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ABSTRACT

In this paper we give new lower bounds on the independence number of a graph in terms of order, maximum degree and girth. We will see that our lower bounds improve (totally or partially) the existing bounds of Shearer, Staton, Hopkins and Staton, Lauer and Wormald.

1. Introduction, notation

We deal with undirected graphs G (graphs for short), without loops and multiple arcs. The vertex set is $V(G)$ and $E(G)$ is the edge set of G. For a vertex x, a neighbor of x is a vertex y such that xy is an edge of G (and then we say that x is adjacent with y). The set of the neighbors of x is $N_G(x)$ and the degree $d_G(x)$ of x is the cardinality of $N_G(x)$. When no confusion is possible, we omit the subscript G. For a set A of vertices of G, $N_A(x)$ is the set of the neighbors of x which are in A.

A cycle of G of length $m \geq 3$ is a sequence $(x_1, \ldots, x_m, x_1)$ of vertices of G such that $x_1, \ldots, x_m$ are distinct, and $x_i, x_{i+1} \in E(G)$ for $1 \leq i \leq m$ (with the convention $x_{m+1} = x_1$). For $m \geq 3$, a m-cycle is a cycle of length m. A 3-cycle is a triangle, and a triangle-free graph is a graph without triangles. For a graph G containing cycles, the girth $g(G)$ of G is the length of a shortest cycle of G. For a graph G without cycles (that is for a forest), we admit that $g(G) = +\infty$.

An independent set of G is a set S of vertices such that any two distinct vertices of G are non-adjacent. The independence number $\alpha(G)$ of G is the size of a largest independent set of G. When no confusion is possible, as for the girth, we omit the parenthesis (G). Several lower bounds on the independence number of triangle-free graphs were given in terms of order, maximum degree and girth.

In 1991, Shearer proved in [3], the following theorems:

Theorem 1.1. *Let G be a triangle-free graph on n vertices with degree sequence $d_1, \ldots, d_n$. Let α be the independence number of G. Let $f(0) = 1, f(d) = \frac{1+(d^2-d)f(d-1)}{d^2+1}$, $d \geq 1$. Then $\alpha \geq \sum_{i=1}^{n} f(d_i)$.*

Theorem 1.2. *Let G be a triangle-free graph on n vertices with degree sequence $d_1, \ldots, d_n$. Suppose that G contains no 3-cycles or 5-cycles. Let n_{11} be the number of pairs of adjacent vertices of degree 1 in G. Let $f(0) = 1, f(1) = \frac{4}{7}, f(d) = \frac{1+(d^2-d)f(d-1)}{d^2+1}$, $d \geq 2$. Then the independence number of G, $\alpha \geq \sum_{i=1}^{n} f(d_i) - \frac{n_{11}}{7}$.*

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Theorem 1.3. *Let G be a graph on n points. Let the vertices have degrees d_i and weights ω_i, $\omega_i \geq 1$. Suppose that G has weighted girth $\geq g$. Let $f(0) = 1, f(1) = \frac{1}{2}, f(2) = \frac{1}{2} - \frac{2}{g}, f(d) = \frac{1+(d^2-d)f(d-1)}{d^2+1}$, $d \geq 3$. Then $\alpha(G) \geq \sum_{i=1}^{n} \left[f(d_i) - (\omega_i - 1)\frac{2}{g}\right]$.*

These theorems imply that for a d-regular graph it holds $\alpha(G) \geq nf(d)$, where $f(m)$, $m \in \mathbb{N}$, is the sequence of one of the three theorems. In particular for cubic and 4-regular graphs, the theorems of Shearer imply:

Theorem 1.4. *Let G be a triangle free graph of independence number α.*

(a) *If G is cubic, we have $\alpha \geq \frac{17}{50}n$.*
(b) *If G is 4-regular, we have $\alpha \geq \frac{127}{425}n$.*

Theorem 1.5. *Let G be a triangle-free graph on n vertices. Suppose that G contains no 3-cycles or 5-cycles.*

(a) *If G is cubic, we have $\alpha \geq \frac{5}{14}n$.*
(b) *If G is 4-regular, we have $\alpha \geq \frac{74}{238}n$.*

Theorem 1.6. *Let G be a triangle-free graph on n vertices and of girth g.*

(a) *If G is cubic, we have $\alpha \geq \frac{2g-6}{5g}n$.*
(b) *If G is 4-regular, we have $\alpha \geq \frac{29g-72}{85g}n$.*

A celebrated result of Staton (see [4]) states:

Theorem 1.7. *Every triangle free graph on n vertices with maximum degree at most three has an independent set of size at least $\frac{5n}{14}$.*

Hopkins and Staton, proved in 1982 (see [1]):

Theorem 1.8. *The independence ratio of a graph with maximum degree Δ and girth at least 6 is at least $\frac{2\Delta-1}{\Delta^2+2\Delta-1}$.*

These same authors proposed in the same paper the three following results:

Theorem 1.9. *If G of order n is cubic and contains no 3-cycle and no 5-cycle, then $\alpha(G) \geq \frac{19}{52}n$.*

Theorem 1.10. *If G of order n is cubic and contains no 3-cycle, no 5-cycle, and no 7-cycle, then $\alpha(G) \geq \frac{20}{53}n$.*

Theorem 1.11. *There exists a sequence $\{\epsilon_k\}$ of positive numbers converging to zero such that if G is a cubic graph with girth bigger than $4k+1$, then $\alpha(G) \geq (\frac{7}{18} - \epsilon_k)n$.*

The last result we consider is that of Lauer and Wormald (see [2]):

Theorem 1.12. *Let G be a d-regular graph of order n, of girth g and of independence number α. Then $\alpha \geq \frac{1}{2}\left[1 - (d-1)^{\frac{-2}{d-2}} - \epsilon(g)\right]n$, where $\epsilon(g)$ tends to 0 as g tends to ∞.*

In our paper, we prove the following:

Theorem 1.13. *Let G be a graph of order n, of maximum degree $\Delta \geq 2$, of girth $g \geq 4$, and of independence number α.*

(a) *If $g \equiv 0 \pmod 4$ or if $g \equiv 1 \pmod 4$, we have $\alpha \geq \frac{2(\Delta-1)^{\lfloor \frac{g}{4} \rfloor}-2}{(\Delta+2)(\Delta-1)^{\lfloor \frac{g}{4} \rfloor}-4}n$.*
(b) *If $g \equiv 2 \pmod 4$ or if $g \equiv 3 \pmod 4$, we have $\alpha \geq \frac{2(\Delta-1)^{\lfloor \frac{g}{4} \rfloor+1}-\Delta}{(\Delta+2)(\Delta-1)^{\lfloor \frac{g}{4} \rfloor+1}-2\Delta}n$.*

We will see that in many cases, our lower bounds improve (partially or not) the other given bounds.

2. Proof of Theorem 1.13

Proof of Theorem 1.13(a). Let us put $k = \lfloor \frac{g}{4} \rfloor$. We have to prove that $\alpha \geq \frac{2(\Delta-1)^k-2}{(\Delta+2)(\Delta-1)^k-4}n$.

Let S be a maximum independent set of G. So, the cardinality of S is α. We recursively define the sets A_i and B_i, in the following way: A_1 is the set of the vertices of $V(G) \setminus S$ having exactly one neighbor in S and B_1 is the set of the neighbors in S of the vertices of A_1. When $k \geq 2$, for $2 \leq i \leq k$, A_i is the set of the vertices of $(V(G) \setminus S) \setminus (A_1 \cup \cdots \cup A_{i-1})$ having exactly