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Rooted cyclic permutations of lattice paths and uniform partitions

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ABSTRACT

A partition of a given set is said to be uniform if all the partition classes have the same cardinality. In this paper, we will introduce the concepts of rooted n-lattice path and rooted cyclic permutation and prove some fundamental theorems concerning the actions of rooted cyclic permutations on rooted lattice n-paths. The main results obtained have important applications in finding new uniform partitions. Many uniform partitions of combinatorial structures are special cases or consequences of our main theorems.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ the set of real numbers. An n-lattice path is a sequence L of two-dimensional vectors

$$L = (x_i, y_i)_{i=1}^{n} = (x_1, y_1)(x_2, y_2) \cdots (x_n, y_n),$$

where $(x_i, y_i) \in \mathbb{N} \times \mathbb{R}$ for every i. We call the vector (x_i, y_i) a step of the path L.

Let $a_0 = 0$, $s_0 = 0$, and

$$a_i = \sum_{j=1}^{i} x_j, \qquad s_i = \sum_{j=1}^{i} y_j, \quad 1 \le i \le n,$$

then L corresponds to the following sequence of points

$$(a_0, s_0)(a_1, s_1) \cdots (a_n, s_n).$$

We call (a_n, s_n) the end point of L. Let $P(L)$ be the set of indices of points above x-axis in the path L, i.e., $P(L) = \{i \mid s_i > 0\}$, and $p(L) = |P(L)|$. Moreover, we use $m(L)$ to denote the index of the leftmost point whose y-coordinate is maximal in the path L, i.e.,

$$m(L) = \min\{i \mid s_i \ge s_j \text{ for any } j \in \{0, 1, \ldots, n\}\}.$$

Example 1.1. Let $L = (2, -1)(1, 2)(1, -1)(1, 1)(1, 1)(1, -1)(2, 1)(2, -1)$. We draw the lattice path which starts at the origin $(0, 0)$ as given in Fig. 1.

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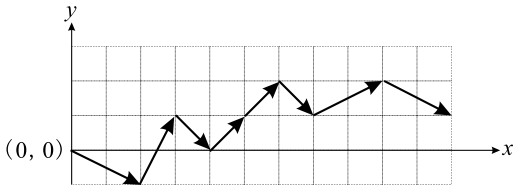

Fig. 1. The lattice path L.

Then

$$P(L) = \{2, 4, 5, 6, 7, 8\}, p(L) = 6 \quad \text{and} \quad m(L) = 5.$$

Given an $i \in \{1, 2, \ldots, n\}$, the ith cyclic permutation L_i of L is the following sequence

$$L_i = (x_i, y_i) \cdots (x_n, y_n)(x_1, y_1) \cdots (x_{i-1}, y_{i-1}).$$

Let

$$\mathcal{P}(L) = \{p(L_i) \mid i \in \{1, 2, \ldots, n\}\},$$
$$\mathcal{M}(L) = \{m(L_i) \mid i \in \{1, 2, \ldots, n\}\}.$$

Obviously, if $s_n > 0$ then

$$\mathcal{P}(L) \subseteq \{1, 2, \ldots, n\},$$
$$\mathcal{M}(L) \subseteq \{1, 2, \ldots, n\}.$$

For every $k \in \{1, 2, \ldots, n\}$, if

$$\mathcal{P}(L) = \{1, 2, \ldots, n\}$$

then there is exactly one cyclic permutation L_i of L such that $p(L_i) = k$, and if

$$\mathcal{M}(L) = \{1, 2, \ldots, n\}$$

then there is exactly one cyclic permutation L_j of L such that $p(L_j) = k$. Thus, for $s_n > 0$, an interesting problem is:

Problem 1.2. Determine necessary and sufficient conditions for

$$\mathcal{P}(L) = \{1, 2, \ldots, n\},$$
$$\mathcal{M}(L) = \{1, 2, \ldots, n\},$$

or

$$\mathcal{P}(L) = \mathcal{M}(L) = \{1, 2, \ldots, n\}.$$

For $s_n \leq 0$, we have

$$\mathcal{P}(L) \subseteq \{0, 1, \ldots, n-1\},$$
$$\mathcal{M}(L) \subseteq \{0, 1, \ldots, n-1\}.$$

Similarly, we are also interested in the following problem:

Problem 1.3. Determine necessary and sufficient conditions for

$$\mathcal{P}(L) = \{0, 1, \ldots, n-1\},$$
$$\mathcal{M}(L) = \{0, 1, \ldots, n-1\},$$

or

$$\mathcal{P}(L) = \mathcal{M}(L) = \{0, 1, \ldots, n-1\}.$$

These problems have been studied by several authors and partial results obtained. In the case of $s_n = 0$, Spitzer [24] gave sufficient conditions for $\mathcal{P}(L) = \mathcal{M}(L) = \{0, 1, \ldots, n-1\}$. Let $L = (x_i, y_i)_{i=1}^n$ be an n-lattice path with $s_n = 1$, where y_i is integer for any $i \in \{1, 2, \ldots, n\}$. Raney [23] discovered a fact: there exists a unique cyclic permutation L_i of L such that $p(L_i) = n$. Narayana [21] gave sufficient conditions for $\mathcal{P}(L) = \{1, 2, \ldots, n\}$. Graham and Knuth's book [12] introduced a simple geometric argument of the results obtained by Raney. This geometric argument gave sufficient conditions for $\mathcal{P}(L) = \mathcal{M}(L) = \{1, 2, \ldots, n\}$. Recently, Huang, Ma and Yeh [14] solved Problems 1.2 and 1.3 completely.