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# The clique-transversal number of a $\{K_{1,3}, K_4\}$-free 4-regular graph[☆]

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ABSTRACT

A clique of a graph $G$ is a complete subgraph maximal under inclusion and having at least two vertices. A clique-transversal set $D$ of a graph $G$ is a set of vertices of $G$ such that $D$ meets all cliques of $G$. The clique-transversal number, denoted by $\tau_c(G)$, is the cardinality of a minimum clique-transversal set in $G$. Wang et al. (2014) proved that $\tau_c(G) = \lceil \frac{n}{3} \rceil$ for any 2-connected $\{K_{1,3}, K_4\}$-free 4-regular graph of order $n$, and conjectured that $\tau_c(G) \leq \frac{10n+3}{27}$ for a connected $\{K_{1,3}, K_4\}$-free 4-regular graph of order $n$.

In this paper, we give a short proof of the aforementioned theorem of Wang et al. and show that the above conjecture is true, apart from only three exceptions.



## 1. Introduction

All graphs considered in this paper are finite and simple. For notations and terminology not defined here, we refer to [6]. For a graph $G = (V, E)$ with vertex set $V$ and edge set $E$, $|V|$ and $|E|$ are its *order* and *size*, respectively. For a vertex $v \in V(G)$, the *neighborhood* $N(v)$ of $v$ is defined as the set of vertices adjacent to $v$. The *degree* of $v$ is equal to $|N(v)|$, denoted by $d_G(v)$. For a nonnegative integer $k$, $G$ is called *$k$-regular* if $d_G(v) = k$ for all $v \in V(G)$. 3-regular graphs are also referred to as *cubic* graphs. A 2-*factor* of $G$ is a 2-regular spanning subgraph of $G$.

A subset $M \subseteq E(G)$ is called a *matching* if no two edges of $M$ have a common end vertex. The vertices incident to the edges of a matching $M$ are *saturated* by $M$; the others are *unsaturated*. A *perfect matching* in a graph is a matching that saturates every vertex. A matching with maximum cardinality among all matchings in $G$ is called a *maximum matching* of $G$. The *matching number*, denoted by $\alpha'(G)$, is the cardinality of a maximum matching in $G$. An *edge covering* of $G$ is a set $L$ of edges such that every vertex of $G$ is incident to some edge of $L$. An edge covering with minimum cardinality among all edge coverings in $G$ is called a *minimum edge covering* of $G$. The *edge covering number* of a graph $G$, denoted by $\beta'(G)$, is the cardinality of a minimum edge covering of $G$.

As usual, the complete graph and the cycle of order $n$ are denoted by $K_n$ and $C_n$, respectively. Let $F_1, \ldots, F_k$ be some graphs, where $k$ is a positive integer. The symbol $F \in \{F_1, \ldots, F_k\}$ represents that $F \cong F_i$ for some $i$. We say that a graph $G$ is *$\{F_1, \ldots, F_k\}$-free* if it does not contain $F_i$ as an induced subgraph for all $i$. For instance, a *triangle-free* graph means $\{K_3\}$-free graph, and a claw-free graph means $\{K_{1,3}\}$-free graph, where $K_{1,3}$ is the complete bipartite graph with parts of cardinalities 1 and 3. In a graph $G$ with at least one cycle, the *girth*, denoted by $g(G)$, is the length of a shortest cycle of $G$. The *line graph* $L(G)$ of $G$ is the graph with the vertex set $E(G)$, in which two vertices are adjacent if and only if they are adjacent as edges in $G$. It is well known that every line graph is claw-free [3]. Since there does not exist four pairwise adjacent edges in a

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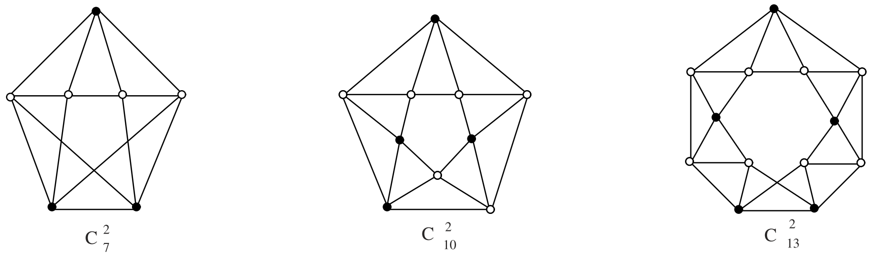


**Fig. 1.** Only three counterexamples to Conjecture 1.

graph of maximum degree 3, the line graph of a cubic graph is a $\{K_{1,3}, K_4\}$-free 4-regular graph. For a graph $G$, $G^2$ denotes the graph obtained from $G$ by adding edges connecting vertices at distance two. For the cycle $C_n$ of order $n \geq 6$, $C_n^2$ is also a $\{K_{1,3}, K_4\}$-free 4-regular graph.

A *clique* $C$ of a graph $G$ is a complete subgraph maximal under inclusion and having at least two vertices. According to this definition, an isolated vertex is not considered to be a clique here. A set $D \subseteq V$ is called a *clique-transversal set* of $G$ if $D$ meets all cliques of $G$, i.e., $D \cap V(C) \neq \emptyset$ for any clique $C$ of $G$. The *clique-transversal number*, denoted by $\tau_c(G)$, is the cardinality of a minimum clique-transversal set of $G$. The concept of clique-transversal set in graph theory can be regarded as a special case of the transversal set in hypergraph theory [4]. Erdős et al. [7] have proved that the problem of finding a minimum clique-transversal number for a graph is NP-hard. In [7], they also showed that every graph of order $n$ has clique-transversal number at most $n - \sqrt{2n} + \frac{3}{2}$ and observed that $\tau_c = n - o(n)$. Tuza [15], Andreae [1], and Andreae and Flotow [2] established upper bounds on clique-transversal number for chordal graphs.

Shan et al. [13] established sharp lower bounds on the clique-transversal number for connected $k$-regular graphs. Shan and Kang [14] give a lower bound on the clique-transversal number for claw-free 4-regular graphs and characterized the extremal graphs achieving the lower bound. Wang et al. [16] proved that $\tau_c(G) = \lceil \frac{n}{3} \rceil$ for any 2-connected $\{K_{1,3}, K_4\}$-free 4-regular graph $G$ of order $n$ by induction on $n$. We give a short proof of this theorem.

Further, they posed the following conjecture.

**Conjecture 1** (*Wang, Shan and Liang [16]*). *If $G$ is a connected $\{K_{1,3}, K_4\}$-free 4-regular graph, then $\tau_c(G) \leq \frac{10n+3}{27}$.*

The aim of this paper is to solve the conjecture. Indeed, we show that $C_7^2$, $C_{10}^2$, $C_{13}^2$ are only three exceptions for the assertion of Conjecture 1. A minimum clique-transversal set of $C_n^2$ for each $n \in \{7, 10, 13\}$ is shown in Fig. 1.

Before stating our main theorem, we introduce a family of graphs, which was first defined by O and West [12]. Let $B$ be the graph, called a *balloon*, obtained from a complete graph $K_4$ by subdividing an edge of $K_4$. Let $T_1$ be the family of trees such that every non-leaf vertex has degree 3 and all leaves have the same color in a proper 2-coloring. Let $\mathcal{H}_1$ be the family of cubic graphs which are obtained from trees in $T_1$ by identifying each leaf of such a tree with the degree 2 vertex in a copy of $B$.

**Theorem 1.1.** *Let $G$ be a connected $\{K_{1,3}, K_4\}$-free 4-regular graph of order $n$. If $G \notin \{C_7^2, C_{10}^2, C_{13}^2\}$ then $\tau_c(G) \leq \frac{10n+3}{27}$, with equality if and only if $G = L(H)$, where $H \in \mathcal{H}_1$.*

## 2. The proof

The key ingredient for proving Theorem 1.3 is a structural result on $\{K_{1,3}, K_4\}$-free 4-regular graphs obtained very recently by Kang et al. [10].

**Theorem 2.1** (*Kang, Wang and Shan [10]*). *If $G$ is a connected $\{K_{1,3}, K_4\}$-free 4-regular graph, then $G \cong F_t$ for some $t \geq 6$, or $G$ is the line graph of a cubic graph, where $V(F_t) = \{v_1, \ldots, v_t\}$ and $E(F_t) = \{v_i v_{i+1} | \ 1 \leq i \leq t-1\} \cup \{v_j v_{j+2} | \ 1 \leq j \leq t-2\} \cup \{v_1 v_{t-1}, v_1 v_t, v_2 v_t\}$.*

Note that the graph $F_t$ defined as above, is nothing but $C_t^2$. So, Theorem 2.1 can be stated equivalently in the following form.

**Theorem 2.2.** *If $G$ is a connected $\{K_{1,3}, K_4\}$-free 4-regular graph of order $n$, then $G \cong C_n^2$ for $n \geq 6$ or $G$ is the line graph of a cubic graph.*

Biedl et al. [5] proved that for any connected cubic graph of order $n$, $\alpha'(G) \geq \frac{4n-1}{9}$. O and West [12] characterized those graphs attaining the lower bound.

**Theorem 2.3** (*O and West [12]*). *If $G$ is a connected cubic graph of order $n$, then $\alpha'(G) \geq \frac{4n-1}{9}$, with equality if and only if $G \in \mathcal{H}_1$.*

Henning et al. [9] established a sharp lower bound for connected cubic triangle-free graphs.