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On the spectrum and number of convex sets in graphs

Jason I. Brown [a], Ortrud R. Oellermann [b,*]

[a] Department of Mathematics and Statistics, Dalhousie University, Halifax NS, Canada

[b] Department of Mathematics and Statistics, University of Winnipeg, Winnipeg, MB R3B 2E9, Canada

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ABSTRACT

A subset S of vertices of a graph G is g-*convex* if whenever u and v belong to S, all vertices on shortest paths between u and v also lie in S. The g-*spectrum* of a graph is the set of sizes of its g-convex sets. In this paper we consider two problems — counting g-convex sets in a graph, and determining when a graph has g-convex sets of every cardinality (such graphs are said to have the *continuum property*). We show that the problem of counting g-convex sets of a graph whose components have diameter at most 2 is #P-complete, but for the class of cographs these sets can be enumerated in linear time. The problem of determining whether or not the g-convexity of a graph has the continuum property is proven to be NP-complete. While every graph is shown to be an induced subgraph of a graph whose g-convexity possesses the continuum property, graphs with the continuum property are rare since for any fixed $\epsilon \in (0, 1)$ it is shown that almost all n-vertex graphs have a gap in their g-spectrum of size at least $\Omega(n^{1-\epsilon})$. Moreover, it is shown that for almost all graphs, every g-convex set is a clique, from which it follows that the number of g-convex sets in a random graph is at least $n^{c \ln n}$ for some constant c. The graph convexity under discussion fits within the class of alignments on a finite set, namely those set systems on a finite set V that contain the whole set, the empty set, and are closed under intersection. Finite topologies are perhaps the most famous examples of alignments, and our results here are compared and contrasted with what can be said for topologies on a finite set.

1. Introduction

A set C of points in Euclidean space is *convex* if for every pair of points in C the line segment that connects them is also in S. It is known that, in Euclidean space, the intersection of any finite number of convex sets is again convex. These observations gave rise to the definition of abstract convexity spaces. Let V be a finite set and let $\mathcal{M}$ be a collection of subsets of V. Then $\mathcal{M}$ is an *alignment* or *convexity* of V if $\mathcal{M}$ is closed under taking intersections and contains both $\emptyset$ and V. If $\mathcal{M}$ is an alignment of V, then the elements of $\mathcal{M}$ are called *convex sets* and the pair $(V, \mathcal{M})$ is called an *aligned space*. For a set $S \subseteq V$, the *convex hull* of S, denoted by $CH(S)$, is the smallest convex set that contains S. A point x of a convex set X is an *extreme point* of X if $X - \{x\}$ is also convex. An alignment $(V, \mathcal{M})$ is a *convex geometry* if every convex set is the convex hull of its extreme points. For a more extensive treatment of abstract convexity see [24].

It is readily seen that if $(V, \mathcal{M})$ is a convex geometry on a finite set V, then V can be ordered as $v_1, v_2, \ldots, v_{|V|}$ such that $S_i = \{v_i, v_{i+1}, \ldots, v_{|V|}\}$ is convex and v_i is an extreme point of S_i for every $i \in \{1, \ldots, |V|\}$. Thus if $(V, \mathcal{M})$ is a convex geometry on a finite set V, then $\mathcal{M}$ contains a convex set of cardinality i for every $i \in \{0, 1, \ldots, |V|\}$. Convexities on a finite

* Corresponding author.
E-mail addresses: jason.brown@dal.ca (J.I. Brown), o.oellermann@uwinnipeg.ca (O.R. Oellermann).

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set V, that have a convex set of cardinality i for every $i \in \{0, 1, \ldots, |V|\}$, are said to have the *continuum property*. While convex geometries on a finite set have the continuum property, the converse is not true, see for example [19]. This gives rise to the problem of determining whether a given alignment has the continuum property and how prevalent such alignments are. In this paper we focus on alignments with the continuum property and the problem of enumerating the number of convex sets of alignments on finite sets. To this end we define, for a given alignment $\mathcal{M}$ of a finite set V, the *spectrum* to be the set $S_{\mathcal{M}} = \{|X| \mid X \in \mathcal{M}\}$.

Let $V = \{v_1, v_2, \ldots, v_n\}$. Then for any sequence $0 = s_0 < s_1 < s_2 < \cdots < s_k = n$ there is an alignment of V whose spectrum is $\{s_0, s_1, \ldots, s_k\}$. To see this observe that if $X_0 = \emptyset$ and for $1 \leq i \leq k, X_i = \{v_1, v_2, \ldots, v_{s_i}\}$, then $\mathcal{M} = \{X_0, X_1, \ldots, X_k\}$ is an alignment of V whose spectrum is $\{s_0, s_1, \ldots, s_k\}$. Thus any subset of $\{0, 1, \ldots, n\}$ that contains both 0 and n is the spectrum of some alignment and hence for every finite set V there is an alignment of V that has the continuum property.

Moreover, if c is an integer such that $2 \leq c \leq 2^n$, then there is an alignment of V that has c convex sets, for if we arrange the elements of the power set of V in nondecreasing order of magnitude, then the first $c - 1$ sets in such an ordering together with V form an alignment of V. It is evident that the problem of counting the convex sets of an alignment and determining whether or not it has the continuum property is much more interesting if the alignment is derived from some type of combinatorial structure.

Convexities on the points of metric spaces have been of particular interest. Let G be a graph with vertex set V and edge set E. The distance between two vertices u and v of G, denoted by $d(u, v)$, is the length of a shortest u–v path in G if such a path exists and is ∞ if no such path exists. It is well known that d is a metric on V.

The most natural way of extending Euclidean convexity to graphs is as follows: a set C of vertices in a graph G is *g-convex* if, for every pair of points u and v of G, the *geodesic interval*, $I_g[u, v]$, between u and v is a subset of C; where $I_g[u, v]$ is the collection of all vertices that belong to some u–v geodesic, i.e., a shortest u–v path. The *closed interval* $I_g[S]$ of a set $S \subseteq V(G)$ is the union of all sets $I_g[u, v]$ where $u, v \in S$. Thus a set S of vertices is g-convex if and only if $I_g[S] = S$. (We will not restrict ourselves only to connected graphs. Thus g-convex sets may induce disconnected graphs.)

Let $\mathcal{M}_g(G)$ be the collection of all g-convex sets of G. It is not difficult to see that $(V, \mathcal{M}_g(G))$ is a convexity, called the *geodesic-* or *g-convexity*. Graphs for which the collection of g-convex sets form a convex geometry have been characterized in [19]. Characterizations of convex geometries of other graph convexities appear, for example, in [11,16,22]. For the g-convexity a vertex is an extreme point of a convex set S if and only if its neighbourhood in S induces a complete subgraph. Vertices whose neighbourhoods induce complete subgraphs are called *simplicial* vertices.

This paper focuses on two problems: (i) the problem of counting g-convex sets and (ii) the problem of determining the existence and prevalence of graphs for which the g-convexity has the continuum property. We will refer to the spectrum of the g-convexity of a graph as its *g-spectrum* and denote it by $\mathcal{S}_g$. In Section 2 we show that the problem of counting the g-convex sets of graphs, even when restricted to graphs whose components have diameter at most 2, is #P-complete. However, for the class of cographs, that is, those graphs without an induced P_4, we show that the g-convex sets can be enumerated in linear time. In Section 3 we show that the problem of determining whether or not the g-convexity of a graph has the continuum property is NP-complete. Moreover, we prove that every graph is an induced subgraph of a graph whose g-convexity possesses the continuum property. Thus graphs that have the continuum property cannot be characterized in terms of forbidden subgraphs. Nevertheless it is shown that graphs with the continuum property are rare since almost all graphs have a (large) gap in their g-spectrum. We conclude Section 3 by showing that for almost all graphs almost all g-convex sets are cliques. Graphs for which all non-empty proper g-convex sets are cliques are called *clique-convex* graphs. In Section 4 we describe several classes of clique-convex graphs. Among these are the Paley graphs of prime order $p \equiv 1 \pmod 4$. The concluding section of the paper is devoted to some additional observations and open problems, and a comparison of results on alignments, including g-convexities, and finite topologies.

2. Counting *g*-convex sets

In [26] it was shown that the number of subtrees of a tree can be determined in linear time. Since the subtrees of a tree are precisely the g-convex sets of a tree, this result shows that the number of g-convex sets of a tree can be determined in linear time. Thus the number of g-convex sets of forests can also be determined in linear time as their number is the product of the number of g-convex sets of each component. However, for graphs in general, the complexity of counting g-convex sets is # P-complete, as we now proceed to show.

Remark 2.1. Let $G = (V, E)$ be a graph of order n and size m. It was shown in [15] that the closed interval $I_g[S]$ of a set S of vertices can be determined in $O(|S|m)$ time. Since S is g-convex if and only if $|S| = |I_g[S]|$ it can be determined in $O(nm)$ time whether a given set of vertices is convex.

It is known that the problem of counting all independent sets of a graph is #P-complete [23]. Consequently the problem of counting all cliques (that is, sets of vertices that induce complete subgraphs) of a graph is also #P-complete (even when restricted to connected graphs, as the number of cliques in a disconnected graph is the sum of the number of cliques in the components).

For a given graph G let $n_{cl}(G)$ be the number of cliques of G (including the empty clique) and $n_{gc}(G)$ the number of g-convex sets of G (including the empty set and $V(G)$). Before proving the main result of this section we establish the following