Discrete Mathematics 338 (2015) 1154–1162

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Circular chromatic indices of even degree regular graphs

Cheyu Lin [a], Tsai-Lien Wong [a,b], Xuding Zhu [c,*]

[a] Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan
[b] National Center for Theoretical Sciences, Taiwan
[c] Department of Mathematics, Zhejiang Normal University, China

ARTICLE INFO

Article history:
Received 24 March 2014
Received in revised form 29 January 2015
Accepted 30 January 2015
Available online 28 February 2015

Keywords:
Circular chromatic index
Even degree regular graph
Construction
Line graph

ABSTRACT

The *circular chromatic index* of a graph G, written $\chi'_c(G)$, is the minimum r permitting a function $c: E(G) \to [0, r)$ such that $1 \le |c(e) - c(e')| \le r - 1$ whenever e and e' are adjacent. It is known that if $r \in (2 + \frac{1}{k+1}, 2 + \frac{1}{k})$ for some positive integer k, or $r \in (\frac{11}{3}, 4)$, then there is no graph G with $\chi'_c(G) = r$. On the other hand, for any odd integer $n \ge 3$, if $r \in [n, n + \frac{1}{4}]$, then there is a simple graph G with $\chi'_c(G) = r$; if $r \in [n, n + \frac{1}{3}]$, then there is a multigraph G with $\chi'_c(G) = r$. For most reals r, it is unknown whether r is the circular chromatic index of a graph (or a multigraph) or not. In this paper, we prove that for any even integer $n \ge 4$, if $r \in [n, n + 1/6]$, then there is an n-regular simple graph G with $\chi'_c(G) = r$; if $r \in [n, n + 1/3]$, then there is an n-regular multi-graph G with $\chi'_c(G) = r$.

1. Introduction

Given a graph G and a real number r, a circular r-*colouring* of G is a function $c: V(G) \to [0, r)$ such that $1 \le |c(x) - c(y)| \le r - 1$ whenever x and y are adjacent. The *circular chromatic number* of G, written as $\chi_c(G)$, is the infimum of all r such that G admits a circular r-colouring. In [17], it was proved that for any rational $r \ge 2$, there is a finite graph G with $\chi_c(G) = r$. A natural question is to determine the possible values of the circular chromatic number of special classes of graphs.

It is well-known [17,21,22] that for a finite graph G, $\chi_c(G)$ is a rational. Nevertheless, if r is irrational, one may obtain an infinite graph G with $\chi_c(G) = r$ by taking the union $\cup_{i=1}^{\infty} G_i$ of infinite many finite graphs, where $\chi'_c(G_i) = r_i$ and r_i are rationals approaching r from below [2].

In this paper, the graphs we shall construct are finite graphs. So their circular chromatic numbers are always rational. Nevertheless, the classes of graphs we shall consider are union closed. By taking infinite union of the constructed graphs, we obtain infinite graphs whose circular chromatic number can be irrational. For simplicity, the graphs in the statements include infinite graphs, and hence their circular chromatic number may be irrationals.

It is known that for any $r \in [2, 4]$, there is a planar graph G with $\chi_c(G) = r$ [15,20]; for any $r \ge 2$ and any integer g, there is a graph G of girth at least g such that $\chi_c(G) = r$ [19]; for any integer $n \ge 5$ and any $r \in [2, n - 1]$, there is a K_n-minor free graph G with $\chi_c(G) = r$ [11], however, for K_4-minor free graphs G, $\chi_c(G) \in [2, 8/3] \cup \{3\}$ [7].

In this paper, we study the circular chromatic number of line graphs. For a graph or a multigraph G, the *line graph* $L(G)$ is the graph with vertex set $E(G)$ with two edges of G adjacent in $L(G)$ if they share a common end vertex. The *circular chromatic index* $\chi'_c(G)$ is defined by $\chi'_c(G) = \chi_c(L(G))$. Recall that the *chromatic index* $\chi'(G)$ is equal to the chromatic number of its line graph $L(G)$. Thus we have $\chi'(G) - 1 < \chi'_c(G) \le \chi'(G)$, and χ'_c is a refinement of χ'. The circular chromatic indices of graphs

* Corresponding author.
E-mail addresses: tlwong@math.nsysu.edu.tw (T.-L. Wong), xudingzhu@gmail.com (X. Zhu).

http://dx.doi.org/10.1016/j.disc.2015.01.037
0012-365X/

have been studied in many papers [1,3–6,8–10,12–14,16,18]. It is known that for a subcubic multigraph G, either $\chi'_c(G) = 4$ or $\chi'_c(G) \leq 11/3$ [1]; subcubic graphs G of girth at least six have $\chi'_c(G) \leq 7/2$ [10]; graphs G of large girth have $\chi'_c(G)$ close to $\Delta(G)$ [9]; for any integers $k \geq 4$, $1 \leq a \leq k/2$, $p \geq 2a^2 + a + 1$, there is a k-regular graph G with $\chi'_c(G) = k + 1/p$ [3]. The circular chromatic index of snarks has been studied in [1,4–6,14], and the circular chromatic index of Cartesian product graphs has been studied in [18]. Nevertheless, the question as which real numbers are the circular chromatic indices of graphs remains largely open.

Let

$$S = \{\chi'_c(G) : G \text{ is a simple graph}\},$$

and

$$M = \{\chi'_c(G) : G \text{ is a multi-graph}\}.$$

We remark that the graphs in the definitions of S and M include infinite graphs. An interval (a, b) is called a *gap* for S if $a, b \in S$ and $(a, b) \cap S = \emptyset$. We define a gap for M similarly. As mentioned above, $(2 + \frac{1}{k+1}, 2 + \frac{1}{k})$ for $k = 1, 2, \ldots$ and $(11/3, 4)$ are gaps for S and M. We do not know if S or M have other gaps.

Opposite to gaps, an S-*interval* or an M-*interval* is an interval $[a, b]$ contained in S or M, i.e., for any $r \in [a, b]$, there is a graph G with $\chi'_c(G) = r$. Lukot'ka and Mazák [13] proved that $[3, 10/3]$ is an S-interval. In [12], we generalized this result and proved that for any odd integer n, $[n, n+1/3]$ is an M-interval and $[n, n+1/4]$ is an S-interval. In this paper, we extend this result and prove that for any even integer n, $[n, n + 1/3]$ is an M-interval and $[n, n + 1/6]$ is an S-interval.

The proof technique is similar to that in [13,12]. The difference is in the construction of the small pieces (monochromatic networks) that are used in the construction of the desired graphs.

2. Construction of monochromatic networks

The following is the main result of this paper.

Theorem 2.1. *Suppose $n \geq 4$ is an even integer. If $0 \leq \epsilon < 1/6$, then there is an n-regular simple graph G with $\chi'_c(G) = n + \epsilon$; if $0 \leq \epsilon < 1/3$, then there is a n-regular multi-graph G with $\chi'_c(G) = n + \epsilon$.*

As observed before, the graphs in Theorem 2.1 include infinite graphs. However, to prove this theorem, it suffices to consider the case that $r = n + \epsilon$ are rationals. Indeed, to prove Theorem 2.1, we shall construct, for each rational r in the specified range, a finite regular graph G with $\chi'_c(G) = r$. As in [13,12], the graph G is obtained by gluing up small building blocks, called monochromatic networks.

Let p, n be positive integers. A *p-line n-regular monochromatic network* is a triple $N = (G^N, X^N, Y^N)$ such that

- G^N is a graph whose vertices have degree n, except that there are $2p$ vertices of degree 1.
- X^N and Y^N are two disjoint p-tuples of vertices of degree 1 in G^N. The ith element of X^N and Y^N are denoted by x_i^N and y_i^N, respectively. The edges incident with vertices x_i^N and y_i^N are denoted by e_i^N and f_i^N, respectively.
- The graph G^N is n-edge-colourable and in every n-edge-colouring of G^N, for each i, the two edges e_i^N and f_i^N have the same colour.

The p-tuples X^N and Y^N are called the *input* and the *output* of N, respectively. If there is no confusion, we will omit the upper indices in X^N, x_i^N, G^N, etc.

Suppose $0 \leq \epsilon < 1$ and $r = n + \epsilon$. For a circular r-edge-colouring c of G^N, let $\partial(c) = \sum_{i=1}^{p} |c(f_i) - c(e_i)|_r$, where $|x - y|_r = \min\{|x - y|, r - |x - y|\}$. The *$\epsilon$-changeability* of N, denoted by $\Delta_\epsilon(N)$, is defined as the supremum of $\partial(c)$, where the supremum is taken over all the r-edge-colourings c of G^N.

The following result was proved in [12]:

Theorem 2.2. *Let n, q be positive integers and $\epsilon > 0$ be a rational number with $q\epsilon \leq 1$. Assume A is a 1-line n-regular monochromatic network with $\Delta_{\epsilon'}(A) = \epsilon'$ for every $\epsilon' < \epsilon$, and B is a 2-line n-regular monochromatic network with $\Delta_{\epsilon'}(B) = q\epsilon' < 1$ for every $\epsilon' < \epsilon$. Moreover, there is an integer $0 \leq u \leq q$ such that the following is true: For any $\delta \in [0, 1)$, there is an $(n + \epsilon)$-edge-colouring φ_δ of B such that $\varphi_\delta(e_1) = 0$, $\varphi_\delta(e_2) = \delta\epsilon$, $\varphi_\delta(f_1) = (\delta + u)\epsilon$ and $\varphi_\delta(f_2) = (q - u)\epsilon$. Then there is an n-regular graph G with $\chi'_c(G) = n + \epsilon$.*

We remark that Theorem 2.2 is proved for the case that n is odd. The same proof works for the case that n is even, with one step needs a small modification. For the proof of Theorem 2.2 in [12], we first construct a graph G with $\chi'_c(G) = r$, where each vertex of G has degree n, except that some vertices has degree 2. Moreover, there is a circular r-edge-colouring of G such that, for each of these degree 2 vertices v, the colours assigned to the two edges incident to v are distance 1 apart. To obtain a n-regular graph G' with $\chi'_c(G) = r$, we take the disjoint union of $n - 1$ copies of G, for each degree 2 vertex v of G, add edges between the $n - 1$ copies of v's to form a copy K_{n-1}. Note that K_{n-1} is Type-1 when n is odd. For n is even, instead of K_{n-1}, we use a Type-1 $(n - 2)$-regular graph, say $K_{n-2,n-2}$. Take the disjoint union of $2(n - 2)$ copies of G, for each degree