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A classification of regular Cayley maps with trivial Cayley-core for dihedral groups

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ABSTRACT

Let $\mathcal{M} = \mathrm{CM}(G, X, p)$ be a regular Cayley map for the finite group G, and let $\mathrm{Aut}^+(\mathcal{M})$ be the orientation-preserving automorphism group of $\mathcal{M}$. Then G can be regarded as a subgroup of $\mathrm{Aut}^+(\mathcal{M})$ in the sense that G acts on itself by left multiplication. The core of G in $\mathrm{Aut}^+(\mathcal{M})$ is called the Cayley-core of $\mathcal{M}$. In this paper, the regular Cayley maps with trivial Cayley-core for dihedral groups are classified. This work is a partial result for our long term goal to classify all regular Cayley maps for dihedral groups.

1. Introduction

Throughout this paper, all graphs are finite, simple and undirected. We use 1 to denote the integer 1 as well as the identity of any group, and use $\mathbf{Z}_n$ to denote the integer residue ring modulo n, the cyclic group of order n, and the set of nonnegative integers smaller than n. We believe that the reader can make a distinction according to the context.

Let G be a finite group, and let X be a *Cayley subset* of G, that is, X is a generating set of G which contains the inverse of each of its elements, but does not contain the identity element of G. A *Cayley graph* $\Gamma = \mathrm{C}(G, X)$ for the pair (G, X) is a graph with vertex set G and dart (ordered pair of adjacent vertices) set $\{(g, gx) \mid g \in G, x \in X\}$. Let p be a cyclic permutation on X. *A Cayley map* $\mathrm{CM}(G, X, p)$ is a 2-cell embedding of the Cayley graph $\mathrm{C}(G, X)$ into an orientable surface with the same cyclic rotation scheme induced by p at each vertex, in other words, for each $x \in X$ the arc $(g, p(x))$ immediately follows the arc (g, x) when traveling around the vertex g on the surface in the given orientation. Since X is closed under taking inverses, for each $x \in X$ there exists a non-negative integer i such that $p^i(x) = x^{-1}$. Define a function $\chi : X \to \mathbf{Z}_{|X|}$ by $\chi(x) = i$ for each $x \in X$, where i is the smallest non-negative integer such that $p^i(x) = x^{-1}$. We call χ the *distribution of inverses* of the Cayley map $\mathrm{CM}(G, X, p)$. A Cayley map $\mathrm{CM}(G, X, p)$ that satisfies the condition $p^t(x^{-1}) = p(x)^{-1}$ for all $x \in X$ is said to be t-balanced (see [11,3]). Particularly, it is said to be balanced if $t = 1$ and antibalanced if $t = -1$ (see [14,15]). An automorphism of a Cayley map $\mathcal{M}$ is a permutation of the set of darts of $\mathcal{M}$ which preserves the incidence relation of the vertices, edges and faces of the map. It is well known that the group of all orientation-preserving automorphisms of $\mathcal{M}$, denoted by $\mathrm{Aut}^+(\mathcal{M})$, always acts semi-regularly on the set of darts of $\mathcal{M}$, that is, the stabilizer in $\mathrm{Aut}^+(\mathcal{M})$ of each dart of $\mathcal{M}$ is trivial. If the action of $\mathrm{Aut}^+(\mathcal{M})$ on the darts of $\mathcal{M}$ is transitive (and therefore regular), we say that the Cayley map $\mathcal{M}$ is a regular Cayley map. For more information about Cayley maps and regular Cayley maps, see [1,13,5,2].

A main tool for studying regular Cayley maps is the general theory of skew-morphisms introduced in [5], and developed further in [3]. Let G be a finite group, $\varphi : G \to G$ a permutation of G of order n (in the full symmetric group $\mathrm{Sym}(G)$), and

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$\pi : G \to \mathbf{Z}_n$ a function from G to the set of nonnegative integers smaller than n. We say that φ is a skew-morphism of G, with associated power function π, if

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b), \quad \text{for all } a, b \in G$$

where $\varphi^{\pi(a)}(b)$ is the image of b under φ applied $\pi(a)$ times. It was proved in [5] that a Cayley map $\mathrm{CM}(G, X, p)$ is regular if and only if there exists a skew-morphism φ of G such that $\varphi(x) = p(x)$ for all $x \in X$. Therefore, regular Cayley maps of a group G are in a one-to-one correspondence with the orbits of skew-morphisms of G that are closed under taking inverses and generate G. Hence knowing all the skew-morphisms of the finite group G allows one to construct a complete list of regular Cayley maps on G.

For convenience, we sometimes use $\mathrm{CM}(G, X, \varphi)$ to denote the regular Cayley map $\mathrm{CM}(G, X, p)$ with the corresponding skew-morphism φ (with restriction to X equal to p). Let π be the power function of φ. It is known that the value of π is 1 for each element of G if $\mathrm{CM}(G, X, \varphi)$ is balanced, and π assumes only two values t and 1 in $\mathbf{Z}_{|X|}$ if $\mathrm{CM}(G, X, \varphi)$ is t-balanced (see [3]). Recently, the author [18] gave a generalized concept as follows.

Definition 1.1. Let $\mathcal{M} = \mathrm{CM}(G, X, \varphi)$ be a regular Cayley map and let π be the power function of φ. Then $\mathcal{M}$ is called of skew-type i if π assumes exactly i values in $\mathbf{Z}_{|X|}$.

Let $\mathcal{M} = \mathrm{CM}(G, X, \varphi)$ be a regular Cayley map and π be the power function of φ. It is well known that the left regular representation L_G of G is a subgroup of $\mathrm{Aut}^+(\mathcal{M})$, and $\mathrm{Aut}^+(\mathcal{M})$ has a complementary factorization as $\mathrm{Aut}^+(\mathcal{M}) = L_G\langle\rho\rangle$ where $\langle\rho\rangle$ is the stabilizer of 1 in $\mathrm{Aut}^+(\mathcal{M})$. For convenience, we identify L_G with G throughout this paper. In other words, we regard G as a subgroup of $\mathrm{Aut}^+(\mathcal{M})$ in the sense that G acts on itself by left multiplication. With the above notations, we define the concept *Cayley-core* as follows.

Definition 1.2. Let $\mathcal{M} = \mathrm{CM}(G, X, \varphi)$ be a regular Cayley map. Then the core of G in $\mathrm{Aut}^+(\mathcal{M})$ is called the Cayley-core of $\mathcal{M}$.

It is well known that [5] the set $\mathrm{Ker}\,\varphi = \{a \in G | \pi(a) = 1\}$ is a subgroup of G. Usually, $\mathrm{Ker}\,\varphi$ does not have to be preserved by φ. The following lemma shows that the Cayley-core of a skew-morphism is exactly the maximal subgroup of $\mathrm{Ker}\,\varphi$ preserved set-wise by φ.

Lemma 1.3 (*[19]*). *Let $\mathcal{M} = \mathrm{CM}(G, X, \varphi)$ be a regular Cayley map. Then the set $\mathrm{Core}\,\varphi := \{h \in G | \pi\big(\varphi^i(h)\big) = 1, i = 0, 1, 2, \ldots\}$ is a normal subgroup of G. Furthermore, $\mathrm{Core}\,\varphi$ coincides with the Cayley-core of $\mathcal{M}$, that is, $\mathrm{Core}\,\varphi = \bigcap_{a\in Aut^+(\mathcal{M})} aGa^{-1}$ if G is regarded as a subgroup of $Aut^+(\mathcal{M})$.*

A major problem in the theory of Cayley maps is classifying the regular Cayley maps for a given class of finite groups. As far as we know, the only complete classification is the regular Cayley maps for cyclic groups, given by Conder and Tucker [4] recently. Partial classifications include classifications of regular balanced Cayley maps for dihedral and generalized quaternion groups [17]; regular anti-balanced Cayley maps for abelian groups [3]; regular t-balanced Cayley maps for dihedral groups [9], dicyclic groups [10], and semidihedral groups [12]; regular Cayley maps of skew-type 3 for dihedral groups [19]; and regular Cayley maps for dihedral group of order twice an odd number [8]. In this paper, the regular Cayley maps with trivial Cayley-core for dihedral groups are classified. This work is a partial result for our long term goal to classify all regular Cayley maps for dihedral groups. We remark that the underlying graphs of regular Cayley maps for dihedral groups are symmetric dihedrants admitting an arc-regular group of automorphisms. Partial classifications of such graphs were given by I. Kovács et al. [7,8], but we do not need to use that classification here.

Now, we introduce the main theorem of this paper which will be proved in Section 4.

Theorem 1.4. *Let $D_{2n} = \langle a, b | a^n = b^2 = (ab)^2 = 1\rangle$ be the dihedral group of order $2n$ and $\mathcal{M} = \mathrm{CM}(D_{2n}, X, p)$ be a regular Cayley map with trivial Cayley-core. Then, up to isomorphism, $n = 3$ and $p = (ab, a^{-1}b, a, a^{-1})$; or $n = 2m$ for an odd $m \geq 3$ and*

$$p = (a^m, ab, a^{2+m}, a^3b, \ldots, a^{m-2}b, a^{2m-1}, a^m b, a, a^{2+m}b, a^3, \ldots, a^{m-2}, a^{2m-1}b).$$

Remark. It is straightforward to check that $\varphi = (1)(b)(ab, a^{-1}b, a, a^{-1})$ is a skew-morphism of D_6 with associated power function π defined as

$$\pi(ab) = \pi(1) = 1, \qquad \pi(a^{-1}) = \pi(a^{-1}b) = 2 \quad \text{and} \quad \pi(a) = \pi(b) = 3.$$

Moreover, $\mathrm{Ker}\,\varphi = \{1, ab\}$ and $\mathrm{Core}\,\varphi = \{1\}$. Therefore, for the case $n = 3$ and $p = (ab, a^{-1}b, a, a^{-1})$, the Cayley map $\mathcal{M} = \mathrm{CM}(D_{2n}, X, p)$ is indeed a regular Cayley map with trivial Cayley-core. As for the remaining cases, we will give the details in Section 3.

The paper is organized as follows. In the next section, we introduce some preliminary lemmas for later use. In Section 3, the regularity of the resulting maps is proved. In the last section, the proof of Theorem 1.4 is given.