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Note

Connections between conjectures of Alon–Tarsi, Hadamard–Howe, and integrals over the special unitary group

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We show the Alon–Tarsi conjecture on Latin squares is equivalent to a very special case of a conjecture made independently by Hadamard and Howe, and to the non-vanishing of some interesting integrals over SU(n). Our investigations were motivated by geometric complexity theory.

1. Introduction

We first describe the conjectures of Alon–Tarsi, Hadamard–Howe, integrals over the special unitary group, and a related conjecture of Foulkes. We then state the equivalences (Theorem 1.9) and prove them.

1.1. Combinatorics I: The Alon–Tarsi conjecture

Call an $n \times n$ array of natural numbers a *Latin square* if each row and column consists of $[n] := \{1, \dots, n\}$. Each row and column of a Latin square defines a permutation σ of n, where the ordered entries of the row (or column) are $\sigma(1), \dots, \sigma(n)$. Define the sign of the row/column to be the sign of this permutation. Define the *column sign* of the Latin square to be the product of all the column signs (which is 1 or -1, respectively called *column even* or *column odd*), the *row sign* of the Latin square to be the product of the row signs and the *sign* of the Latin square to be the product of the row sign and the column sign.

Conjecture 1.1 ([1] Alon–Tarsi)**.** *Let n be even, then the number of even Latin squares of size n does not equal the number of odd Latin squares of size n.*

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Conjecture 1.1 is known to be true when $n = p \pm 1$, where p is an odd prime; in particular, it is known to be true up to $n = 24$ [10,8].

The Alon–Tarsi conjecture is known to be equivalent to several otherconjectures in combinatorics. For our purposes, the most important is the following due to Huang and Rota:

Conjecture 1.2 (*[16] Column-sign Latin Square Conjecture*). *Let n be even, then the number of column even Latin squares of size n does not equal the number of column odd Latin squares of size n.*

Theorem 1.3 (*[16, Identities 8, 9]*). *The difference between the number of column even Latin squares of size n and the number of column odd Latin squares of size n equals the difference between the number of even Latin squares of size n and the number of odd Latin squares of size n, up to sign. In particular, the Alon–Tarsi conjecture holds for n if and only if the column-sign Latin square conjecture holds for n.*

Remark 1.4. It is easy to see that for n odd, the number of even Latin squares of size n equals the number of odd Latin squares of size n.

1.2. The Hadamard–Howe conjecture

Let V be a finite dimensional complex vector space, let $V^{\otimes n}$ denote the space of multi-linear maps $V^* \times \cdots \times V^* \to \mathbb{C}$, the space of *tensors*. The permutation group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the inputs of the map. Let $S^nV \subset V^{\otimes n}$ denote the subspace of symmetric tensors, the tensors invariant under $\mathfrak{S}_n$, which we may also view as the space of homogeneous polynomials of degree n on V^*. *We will always view S^nV as the subspace of $V^{\otimes n}$ consisting of the symmetric tensors.* In particular, for $v_i \in V$, the notation

$$v_1 \cdots v_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \in S^nV.$$

Let $\mathrm{Sym}(V) := \oplus_d S^dV$, which is an algebra under multiplication of polynomials. Let $\mathrm{GL}(V)$ denote the general linear group of invertible linear maps $V \to V$. Consider the $\mathrm{GL}(V)$-module map

$$h_{d,n} : S^d(S^nV) \to S^n(S^dV)$$

given as follows: Include $S^d(S^nV) \subset V^{\otimes nd}$. Write $V^{\otimes nd} = (V^{\otimes n})^{\otimes d}$, as d groups of n vectors reflecting the inclusion. Now rewrite $V^{\otimes nd} = (V^{\otimes d})^{\otimes n}$ by grouping the first vector space in each group of n together, then the second vector space in each group, etc. Next symmetrize within each group of d to obtain an element of $(S^dV)^{\otimes n}$, and finally symmetrize the groups to get an element of $S^n(S^dV)$.

For example $h_{d,n}((x_1)^n \cdots (x_d)^n) = (x_1 \cdots x_d)^n$ and $h_{3,2}\big((x_1x_2)^3\big) = \frac{1}{4}x_1^3x_2^3 + \frac{3}{4}(x_1^2x_2)(x_1x_2^2)$.

The map $h_{d,n}$ was first considered by Hermite [14] who proved that, when $\dim V = 2$, the map is an isomorphism. It had been conjectured by Hadamard [12] and tentatively conjectured by Howe [15] (who wrote “is reasonable to expect”) that $h_{d,n}$ is always of maximal rank, i.e., injective for $d \leq n$ and surjective for $d \geq n$. A consequence of the theorem of [24] (explained below) is that, contrary to the expectation above, $h_{5,5}$ is not an isomorphism.

For any $n \geq 1$, define the *Chow variety*

$$\mathrm{Ch}_n(V^*) := \{P \in S^nV^* \mid P = \ell_1 \cdots \ell_n \text{ for some } \ell_j \in V^*\}.$$

(This is a special case of a Chow variety, namely of the zero cycles in projective space, but it is the only one that we discuss in this article.) In [4,5], Brion (and independently Weyman and Zelevinsky) observed that $\oplus_d S^n(S^dV)$ is the coordinate ring of the normalization of the Chow variety. (Given an irreducible affine variety Z, its *normalization* $\tilde{Z}$ is an irreducible affine variety whose ring of regular functions is integrally closed and such that there is a regular, finite, birational map $\tilde{Z} \to Z$, see e.g., [27, Chap. II S 5].)

Lemma 1.5 (*Hadamard, See e.g. [20, Section 8.6]*). *The kernel of the* $\mathrm{GL}(V)$*-module map*

$$\oplus h_{d,n} : \mathrm{Sym}(S^nV) := \oplus_d S^d(S^nV) \to \oplus_d S^n(S^dV)$$

is the ideal of the Chow variety.

Brion also showed that for d exponentially large with respect to n, $h_{d,n}$ is surjective [5]. McKay [23] showed that if $h_{d,n}$ is surjective, then $h_{d',n}$ is surjective for all $d' > d$, using $h_{d,n:0}$ defined below. It is also known that if $h_{d,n}$ is surjective, then $h_{n,d}$ is injective, see [17].

The irreducible $\mathrm{GL}(V)$-modules appearing in the tensor algebra of V are indexed by partitions $\pi = (p_1 \geq p_2 \geq \cdots \geq p_q \geq 0)$, $q \leq \dim V$, and denoted $S_\pi V$. If π is a partition of d, i.e., $|\pi| := p_1 + \cdots + p_q = d$, the module $S_\pi V$ appears in $V^{\otimes d}$ and in no other degree. We will use the notation $s\pi := (sp_1, \ldots, sp_q)$. Repeated numbers in partitions are sometimes expressed as exponents when there is no danger of confusion, e.g., $(3, 3, 1, 1, 1, 1) = (3^2, 1^4)$. Let $\mathrm{SL}(V)$ be the subgroup of $\mathrm{GL}(V)$ consisting of determinant 1 elements, and let $\mathfrak{sl}(V)$ denote its Lie algebra.