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Characterization of extremal graphs from Laplacian eigenvalues and the sum of powers of the Laplacian eigenvalues of graphs

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ABSTRACT

For any real number α, let $s_\alpha(G)$ denote the sum of the αth power of the non-zero Laplacian eigenvalues of a graph G. In this paper, we first obtain sharp bounds on the largest and the second smallest Laplacian eigenvalues of a graph, and a new spectral characterization of a graph from its Laplacian eigenvalues. Using these results, we then establish sharp bounds for $s_\alpha(G)$ in terms of the number of vertices, number of edges, maximum vertex degree and minimum vertex degree of the graph G, from which a Nordhaus–Gaddum type result for s_α is also deduced. Moreover, we characterize the graphs maximizing s_α for $\alpha > 1$ among all the connected graphs with given matching number.

1. Introduction

Throughout this paper we consider simple graphs, namely graphs without loops and multiple edges. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|E(G)| = m$. Let d_i denote the degree of vertex v_i in G, $i = 1, 2, \ldots, n$. We also denote by $\Delta(G)$ and $\delta(G)$, respectively, the maximum degree and minimum degree of vertices in G. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix and $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees of G. The Laplacian spectrum of G is

$$\text{Spec}_L(G) = \{\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\},$$

where $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G)$ are the eigenvalues of $L(G)$ (usually known as the Laplacian eigenvalues of G), arranged in non-increasing order. It is known that $\mu_n(G) = 0$ and the multiplicity of 0 is equal to the number of connected components in G. Thus, a graph G is connected if and only if $\mu_{n-1}(G) > 0$, and hence $\mu_{n-1}(G)$ is named the algebraic connectivity of G [5]. For more properties of the Laplacian spectrum of graphs one may refer to [7,15,17].

The Laplacian matrix of a graph and its spectrum can be used in several areas of mathematical, physical and chemical research [17]. Historically, one of the first applications of the Laplacian matrix is in the proof of the well-known Matrix-Tree Theorem which states that the number of spanning trees in a graph is equal to the absolute value of any cofactor of the

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Laplacian matrix of the graph. Another classical application is related to the Kirchhoff index of a connected graph G, which was, originally, defined by Klein and Randić [10] as

$$Kf(G) = \sum_{i<j} r_{ij}\,,$$

where r_{ij} is the resistance distance between the vertices v_i and v_j in G. It is somewhat surprising that this index can be expressed via the non-zero Laplacian eigenvalues [8], i.e.,

$$Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i(G)}.$$

The Kirchhoff index has found a great deal of applications in electric circuit, probabilistic theory and chemistry, and many of its mathematical properties have been established [23].

Recently, a novel Laplacian-spectrum-based graph invariant, known as Laplacian-energy-like invariant, was put forward by Liu and Liu [12]:

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}.$$

The motivation for introducing LEL was in its analogy to the earlier studied graph energy and Laplacian energy. Many mathematical investigations for LEL have been communicated, see [11] for a survey.

As a natural extension of Kf and LEL, Zhou [21] introduced the following graph spectral invariant:

$$s_\alpha(G) = \sum_{i=1}^{h} \mu_i^\alpha(G), \tag{1.1}$$

where α is an arbitrary real number and h is the number of non-zero Laplacian eigenvalues of the graph G. It is obvious that $Kf(G) = n\,s_{-1}(G)$ (for a connected graph G) and $LEL(G) = s_{1/2}(G)$.

Various (upper and lower) bounds on $s_\alpha(G)$ have been established. Zhou [21] first obtained several bounds for $s_\alpha(G)$ in terms of the number of vertices, number of edges, maximum degree and the number of spanning trees of the graph G. Later in [2], Chen and Qian estimated $s_\alpha(G)$ using the number of vertices, connectivity and chromatic number of the graph G. Recently, Das, Xu and Liu [4] established several bounds on $s_\alpha(G)$ in terms of the number of vertices, number of edges, maximum degree, clique number, independence number and the number of spanning trees of the graph G. For more on this aspect we refer the reader to [13,18].

A matching in a graph G is a set of its disjoint edges, and the matching number of G, denoted by $\beta(G)$, is the maximum cardinality of a matching over all its possible matchings. Let $\mathscr{G}_{n,\,\beta}$ be the set of connected graphs with n vertices and matching number β. Zhou and Trinajstić [22] determined the extremal Kirchhoff index of graphs in $\mathscr{G}_{n,\,\beta}$ and characterized the corresponding extremal graphs. Recently, Xu and Das [19] considered similar problems for the Laplacian-energy-like invariant.

The paper is organized as follows. In Section 2, we list some previously known results that will be used in the subsequent sections. In Section 3, we obtain sharp bounds on the largest and the second smallest Laplacian eigenvalues of a graph, and a new spectral characterization of a graph from its Laplacian eigenvalues. Using the results presented in Sections 2 and 3, we establish in Section 4 sharp bounds for $s_\alpha(G)$ in terms of the number of vertices, number of edges, maximum vertex degree and minimum vertex degree of the graph G, from which a Nordhaus–Gaddum type result for s_α is also deduced. Finally, in Section 5, we characterize the graphs maximizing s_α for $\alpha > 1$ among all connected graphs with given matching number.

2. Preliminaries

Let K_n and $K_{n_1,\,n_2}$ $(n_1 + n_2 = n)$, as usual, denote the complete graph and the complete bipartite graph on n vertices, respectively. Denote by $G \cup H$ the vertex-disjoint union of graphs G and H. In particular, kG stands for the vertex-disjoint union of k copies of G. Let $G \vee H$ be the graph obtained from $G \cup H$ by adding all possible edges joining the vertices in G with those in H. Denote by $\overline{G}$ the complement of the graph G. Clearly, $G \vee H \cong \overline{\overline{G} \cup \overline{H}}$.

We now list some known results that will be useful in the subsequent sections.

Lemma 2.1 ([17]). *Let G and H be two graphs with n and n' vertices, respectively. Then*

(i) $\mathrm{Spec}_L\left(\overline{G}\right) = \{n - \mu_{n-1}(G), n - \mu_{n-2}(G), \ldots, n - \mu_1(G), 0\}$;
(ii) $\mathrm{Spec}_L\,(G \cup H) = \{\mu_1(G), \ldots, \mu_{n-1}(G), \mu_1(H), \ldots, \mu_{n'-1}(H), 0, 0\}$.

For a graph G, if its vertex set $V(G)$ can be partitioned into two non-empty subsets U and W such that each vertex in U has degree r and each vertex in W has degree s, then G will be called an $(r,\ s)$-semiregular graph. In particular, if $r = s$ in an $(r,\ s)$-semiregular graph, then G will be called an r-regular graph.