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Uniformly resolvable designs with block sizes 3 and 4

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ABSTRACT

A uniformly resolvable design (URD) is a resolvable design in which each parallel class contains blocks of only one block size k. Such a class is denoted k-pc and for a given k the number of k-pcs is denoted r_k. Let v denote the number of points of the URD. For the case of block sizes 3 and 4 (both existing), the necessary conditions imply that $v \equiv 0 \pmod{12}$. It has been shown that almost all URDs with permissible r_3 and r_4 exist for $v \equiv 0 \pmod{24}$, $v \equiv 0 \pmod{60}$, $v \equiv 36 \pmod{144}$ or $v \equiv 36 \pmod{108}$. In this paper, we prove that the necessary conditions for the existence of a URD with block sizes 3 and 4 are also sufficient, except when $v = 12$, $r_3 = 1$ and $r_4 = 3$.

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1. Introduction

Let v and λ be positive integers, and let K and M be two sets of positive integers. A *group divisible design*, denoted GDD$(K, M; v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of *points*, $\mathcal{G}$ is a partition of X into *groups*, and $\mathcal{B}$ is a collection of subsets of X, called *blocks*, such that

1. $|B| \in K$ for each $B \in \mathcal{B}$,
2. $|G| \in M$ for each $G \in \mathcal{G}$,
3. $|B \cap G| \leq 1$ for each $B \in \mathcal{B}$ and each $G \in \mathcal{G}$, and
4. each pair of elements of X from distinct groups is contained in exactly one block.

If $K = \{k\}$, respectively $M = \{m\}$, then the GDD$(K, M; v)$ is simply denoted GDD$(k, M; v)$, respectively GDD$(K, m; v)$. A GDD$(K, 1; v)$ is called a *pairwise balanced design* and denoted PBD$(K; v)$. A GDD$(k, m; mk)$ is called a *transversal design* and denoted TD(k, m). We usually use an "exponential" notation to describe the multiset M: a K-GDD of type $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size g_i, $i = 1, 2, \ldots, s$.

In a GDD$(K, M; v)(X, \mathcal{G}, \mathcal{B})$ a *parallel class* is a set of blocks, which partitions X. If $\mathcal{B}$ can be partitioned into parallel classes, then the GDD$(K, M; v)$ is said to be *resolvable* and denoted RGDD$(K, M; v)$. Analogously, a resolvable PBD$(K; v)$ is denoted RPBD$(K; v)$. A parallel class is said to be *uniform* if it contains blocks of only one size k (k-pc). If all parallel classes of an RPBD$(K; v)$ are uniform, the design is said to be *uniformly resolvable*. Here, a uniformly resolvable design RPBD$(K; v)$ is denoted URD$(K; v)$. In a URD$(K; v)$ the number of parallel classes with blocks of size k is denoted r_k, $k \in K$.

In [19], Rees introduced the notation of URDs and showed that all admissible URDs$(\{2, 3\}; v)$ exist. For $K = \{2, 4\}$, almost all URD$(K; v)$ have been constructed in [8,25], with a small number of cases unsettled. For $K = \{3, 4\}$, we summarize the known results as follows.

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Theorem 1.1. *The necessary conditions for the existence of a URD*$(\{3, 4\}; v)$ *with* $r_3, r_4 > 0$ *are* $v \equiv 0 \pmod{12}$, r_4 *is odd,* $1 \le r_4 \le \frac{v}{3} - 1$ *and* $r_3 = \frac{v-1-3r_4}{2}$.

Theorem 1.2 ([7,23–25]). *Let* $v \equiv 0 \pmod{12}$. *There exists a URD*$(\{3, 4\}; v)$ *with* $r_4 = 1, 3, 5, 7$ *or* 9, *except for* $v = 12$ *and* $r_4 \in \{3, 5, 7, 9\}$ *or* $v = 24$ *and* $r_4 = 9$.

Theorem 1.3 ([24,25]). *Let* $v \equiv 0 \pmod{12}$. *There exists a URD*$(\{3, 4\}; v)$ *with* $r_3 = 1, 4, 7$ *or* 10, *except for* $v = 12$ *and* $r_3 \in \{1, 7, 10\}$, *and possibly excepting:*

1. $r_3 = 7$ *and* $v \in \{84, 108, 132, 156, 204, 228, 276, 348, 372, 444\}$*; or*
2. $r_3 = 10$ *and* $v \in \{108, 132, 156, 204, 228, 276, 348, 372, 492\}$.

Theorem 1.4 ([24]). *There exist all admissible URDs*$(\{3, 4\}; v)$, $v \equiv 0 \pmod{12}$, $v < 200$, *except when* $v = 12$ *and* $r_4 = 3$ *and possibly excepting:*

1. $v = 84$: $r_4 = 23$;
2. $v = 108$: $r_4 \in \{29, 31\}$;
3. $v = 120$: $r_4 \in \{27, 29, 31\}$;
4. $v = 132$: $r_4 \in \{35, 37, 39\}$;
5. $v = 156$: $r_4 \in \{41, 43, 45, 47\}$.

Theorem 1.5 ([25,24]). *There exist all admissible URDs*$(\{3, 4\}; v)$ *for* $v \equiv 0 \pmod{24}$, *possibly excepting:*

1. $v = 120$ *and* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$;
2. $v = 264$ *and* $r_4 = (v/3) - 9$;
3. $v = 408$ *and* $r_4 \in \{(v/3) - 15, (v/3) - 13, (v/3) - 11, (v/3) - 9\}$;
4. $v = 456$ *and* $r_4 \in \{(v/3) - 11, (v/3) - 9\}$;
5. $v = 552$ *and* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$;
6. $v = 984$ *and* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$;
7. $v = 1128$ *and* $r_4 = (v/3) - 9$;
8. $v = 3288$ *and* $r_4 = (v/3) - 9$.

Theorem 1.6 ([24]). *There exist all admissible URDs*$(\{3, 4\}; v)$ *for* $v \equiv 0 \pmod{60}$.

In this paper, we consider the entire existence problem of the URD$(\{3, 4\}; v)$ and show that the necessary conditions are also sufficient, with only one exception.

Theorem 1.7. *There exists a URD*$(\{3, 4\}; v)$ *with* $r_3, r_4 > 0$ *if and only if* $v \equiv 0 \pmod{12}$, r_4 *is odd and* $1 \le r_4 \le \frac{v}{3} - 1$, *except for* $v = 12$ *and* $r_4 = 3$.

2. Preliminaries

A group divisible design $(X, \mathcal{G}, \mathcal{B})$ is called *frame resolvable* (and is referred to as a *frame*) if its block set $\mathcal{B}$ admits a partition into *holey parallel classes*, each holey parallel class being a partition of $X \backslash H$ for some group $H \in \mathcal{G}$. The groups in a frame are often referred to as *holes*. The *hole type* of a frame is just its group type as a GDD. It is well known that in a k-frame, each hole must have size a multiple of $k - 1$; in fact the number of holey parallel classes with respect to a given hole H is precisely $|H|/(k - 1)$.

Theorem 2.1 ([5,6,10,17,13,18,22,27]). *The necessary conditions for the existence of a k-frame of type* h^u, *namely,* $u \ge k + 1$, $h \equiv 0 \pmod{k - 1}$ *and* $h(u - 1) \equiv 0 \pmod{k}$, *are also sufficient for*

$k = 2$;

$k = 3$; *and for*

$k = 4$, *and possibly excepting:*

1. $h = 36$ *and* $u = 12$;
2. $h \equiv 6 \pmod{12}$ *and*
   (a) $h = 6$ *and* $u \in \{7, 23, 27, 35, 39, 47\}$;
   (b) $h = 18$ *and* $u \in \{15, 23, 27\}$;
   (c) $h \in \{30, 66, 78, 114, 150, 174, 222, 246, 258, 282, 318, 330, 354, 534\}$ *and* $u \in \{7, 23, 27, 39, 47\}$;
   (d) $h \in \{n : 42 \le n \le 11238\} \backslash \{66, 78, 114, 150, 174, 222, 246, 258, 282, 318, 330, 354, 534\}$ *and* $u \in \{23, 27\}$.

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