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On the orientably-regular embeddings of graphs of order prime-cube

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ABSTRACT

This paper characterizes the automorphism group G of the orientably-regular embeddings of simple graphs of order prime-cube p^3. Our main result will be a starting point for classifying all such embeddings. Moreover, by using some known results, a partial classification is given, when G contains a Sylow p-subgroup of order p^5.

1. Introduction

A (topological) *map* is a cellular decomposition of a closed surface. A common way to describe such a map is to view it as a 2-cell embedding of a connected graph or multigraph Γ into the surface S. The components of the complement $S\setminus\Gamma$ are simply-connected regions called the *faces* of the map (or the embedding). An *automorphism* of a map $\mathcal{M}$ is an automorphism of the underlying (multi) graph Γ which can be extended to a self-homeomorphism of the supporting surface S. It is well known that the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a map $\mathcal{M}$ acts semi-regularly on the set of all incident vertex–edge–face triples (or *flags* of Γ). In particular, if $\mathrm{Aut}(\mathcal{M})$ acts regularly on the flags, we call it a *regular map*. In the case of orientable supporting surface, if the group of all orientation-preserving automorphisms of $\mathcal{M}$ acts regularly on the set of all incident vertex–edge pairs (or *arcs*) of $\mathcal{M}$, then we call $\mathcal{M}$ an *orientably-regular* map. Such maps fall into two classes: those that admit also orientation-reversing automorphisms, which are called *reflexible*, and those that do not, which are *chiral*. Therefore, a reflexible map is a regular map but a chiral map is not.

One of the central problems in topological graph theory is to classify all the regular or *orientably-regular* embeddings of a given graph. In a general setting, the classification problem was treated by Gardiner, Nedela, Širáň and Škoviera in [8]. For particular classes of graphs, it has been solved. Here we just mention one family of graphs which will be used in this paper. Let $K_{m[n]}$ be the complete multipartite graph with m parts, while each part contains n vertices. All the *orientably-regular* embeddings of complete graphs $K_{m[1]}$ have been determined by Biggs, James and Jones [1,11], and the regular embeddings of $K_{m[1]}$ on nonorientable surfaces have been classified by Wilson [24]. As for the complete bipartite graphs $K_{2[n]}$, the regular embeddings of these graphs on nonorientable surfaces have been classified by Kwak and Kwon [19]; during the past twenty years, several papers [5,7,12,14,15,17,18,22] contributed to the orientably-regular case, and the final classification was given by Jones [13] in 2010. As for $K_{m[n]}$ where $m \geq 3$, the orientably-regular maps have been classified by Du and Zhang [3,25], where the case when n is a prime has been done in [6], and the nonorientable regular maps have been done by Kwon [20].

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The classification of orientably-regular maps with prime p vertices was achieved in [4], while regular maps of p vertices on nonorientable surfaces was determined in [24]. It has been classified in [6,3,25] that orientably-regular embeddings and regular embeddings of graphs of order a product of pq for any two primes p and q (not necessarily distinct). A natural problem is to determine the orientably-regular embeddings and regular embeddings of graphs of order p^3, where p is prime.

This paper contributes the classification of orientably-regular embeddings of graphs with p^3 vertices. Although underlying graphs of regular maps may have multiple edges, a regular map with multiple edges projects onto another one with a simple underlying graph that has the same set of vertices and the same adjacency relation. Hence, this paper will focus on the orientably-regular embeddings of simple underlying graphs of order p^3. The embeddings of such order with multiple edges will be determined in our further papers.

The main result of this paper is the following theorem which gives a characterization of the automorphism group of the embeddings mentioned as above. Based on this characterization, the complete classification will be given in future.

Theorem 1.1. *Let $\mathcal{G}$ be a connected simple graph of order p^3 where p is prime and let $\mathcal{M}$ be an orientably-regular embedding of $\mathcal{G}$ with the group $G = \langle r, \ell\rangle$ of all orientation-preserving automorphisms, where ℓ is an involution and $\langle r\rangle = G_v$ for a vertex v in $V(\mathcal{G})$. Take a Sylow p-subgroup P of G. Then we have*

(1) $|P| = p^3, p^4$ *or* p^5.
(2) $G = P \rtimes \langle r^m\rangle$ *where* $m = |\langle r\rangle \cap P|$, *except for the case* $p = 2$ *and* $G \cong S_4$.
(3) *Let N be a minimal normal subgroup of G which induces blocks of shortest length and K the kernel of G on the block system. Then one of the following holds:*
   (3.1) *N is transitive on $V(\mathcal{G})$ and G is a primitive affine group.*
   (3.2) $p = 2$, $N = \mathbb{Z}_2^2$ *and* $G = S_4$.
   (3.3) *N induces blocks of length p such that* $N \cong \mathbb{Z}_p \le Z(P)$ *and either* $K \cong \mathbb{Z}_p \rtimes \mathbb{Z}_t$ *for some* $t \in \mathbb{Z}_p^*$; *or* $K \cong \mathbb{Z}_p^2$.

Remark 1.2. Theorem 1.1 is an important starting point for classifying all such embeddings. Since $G = P \rtimes \langle r^m\rangle$, except for a small case $p = 2$ and $G = S_4$, we may conclude the classification problem to the investigation of the split cyclic extension of Sylow p-subgroup P with some extra conditions, where $|P| = p^3, p^4$ and p^5. Equivalently, we just need to determine certain cyclic subgroups of $\mathrm{Aut}(P)$. All the three cases are still quite complicate. Fortunately, for the case $|P| = p^5$, some known results in [6,3,5,7,25,26] can be used and so the classification can be given in this paper, see Theorem 1.3. As for the cases $|P| = p^3$ and $|P| = p^4$, we have to do that in the separate papers.

As usual, an orientably-regular map will be presented by a triple $(G; r, \ell)$ for a group G generated by an element r and an involution ℓ (see Section 2 for the details).

Theorem 1.3. *For a prime p and integer $n \ge 2$, let $\mathcal{G}$ be a connected simple connected graph of order p^3 and valency n. Suppose that $\mathcal{M}$ is an orientably-regular embedding of $\mathcal{G}$, whose group G of all orientation-preserving automorphisms contains a Sylow p-subgroup of order p^5. Then G, $\mathcal{M}$ and the genus g are given by*

(1) $p = 2, n = 4$:

$$G_1 \cong \langle a, b, x \mid a^4 = b^4 = x^2 = 1, [a, b] = 1, a^x = b\rangle,$$
$$\mathcal{M}_1 = \mathcal{M}(G_1; a, x), \quad g = 3.$$

(2) $p = 2, n = 4$:

$$G_2 \cong \langle a, b, x \mid a^4 = b^4 = x^2 = 1, [b, a] = a^2b^2, [a^2, b] = [b^2, a] = 1, a^x = b\rangle,$$
$$\mathcal{M}_2 = \mathcal{M}(G_2; a, x), \quad g = 1.$$

(3) $p = 3, n = 18$:

$$G_3 \cong \langle a, b \mid a^{18} = b^2 = c^{27} = 1, c = a^9b, c^a = c^2\rangle,$$
$$\mathcal{M}_3(j) = \mathcal{M}(G_3; a^j, b), \quad \text{where } j \in \mathbb{Z}_{18}^*, \quad g = 100.$$

(4) $p = 3, n = 18$:

$$G_4(i, j) \cong \langle a, b \mid a^{18} = b^2 = 1, a^2 = x, x^b = y, [x, y] = x^{3i}y^{-3i}, y^a = x^{-1}y^{-1}, (ab)^3 = x^{3j}y^{-3j}\rangle$$

where $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1)$ *or* $(1, -1)$,

$$\mathcal{M}_4(i, j, l) = \mathcal{M}(G_4(i, j); a^l, b),$$

where $l = 1$ *for* $(i, j) = (0, 0)$ *and* $l = \pm 1$ *for the other cases,*
$g = 28$ *for* $(i, j, l) = (0, 0, 1)$ *and* $(1, 0, \pm 1)$;
$g = 82$ *for* $(i, j, l) = (0, 1, \pm 1)$ *and* $(1, \pm 1, \pm 1)$.