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A generalization of Aztec diamond theorem, part II

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ABSTRACT

The author gave a proof of a generalization of the Aztec diamond theorem for a family of 4-vertex regions on the square lattice with southwest-to-northeast diagonals drawn in Lai (2014) by using a bijection between tilings and non-intersecting lattice paths. In this paper, we use Kuo graphical condensation to give a new proof.

1. Introduction

A *lattice* divides the plane into non-overlapped parts called *fundamental regions*. A *region* considered in this paper is a finite connected union of fundamental regions. We define a *tile* to be the union of any two fundamental regions sharing an edge. We are interested in how many different ways to cover a region by tiles such that there are no gaps or overlaps; such coverings are called *tilings*. We use the notation $M(R)$ for the number of tilings of a region R.

The *Aztec diamond* of order n is the union of all the unit squares inside the contour $|x| + |y| = n + 1$. Fig. 1.1 illustrates the Aztec diamonds of order 1, 2, and 4 with a checkboard coloring. The well-known Aztec diamond theorem by Elkies, Kuperberg, Larsen and Propp [4,5] states that the number of (domino) tilings of the Aztec diamond of order n is equal to $2^{n(n+1)/2}$. The first four proofs of the theorem were presented in [4,5], and many further proofs followed (see e.g. [1,2,6–9,12]).

Chris Douglas [3] considered a variant of the Aztec diamond on the square lattice with every second southwest-to-northeast diagonal drawn in. The first three Douglas regions $D(n)$'s are illustrated in Fig. 1.2. More precisely, the four vertices of $D(n)$ (indicated by the dots in Fig. 1.2) are always the vertices of a diamond with side-length $2n\sqrt{2}$. The northwest and the southeast boundaries of $D(n)$ are the same as that of the Aztec diamond of order $2n$, and the northeast and the southwest boundaries are two zigzag paths with steps of length 2. Douglas [3] proved a conjecture posed by Propp that the region $D(n)$ has $2^{2n(n+1)}$ tilings.

We consider a family of regions first introduced in [10], which can be considered as a common generalization of the Aztec diamonds and Douglas regions.

From now on, the term "diagonal" will be used to mean "southwest-to-northeast lattice diagonal". The distances between any two successive drawn-in diagonals of the Douglas region $D(n)$ are all $\sqrt{2}$. Next, we consider the general situation when the distances between two successive drawn-in diagonals are arbitrary.

Suppose we have two lattice diagonals ℓ and ℓ' that are *not* drawn-in diagonals, so that ℓ' is below ℓ. Assume that $k-1$ diagonals have been drawn between ℓ and ℓ', with the distances between successive ones, starting from top, being

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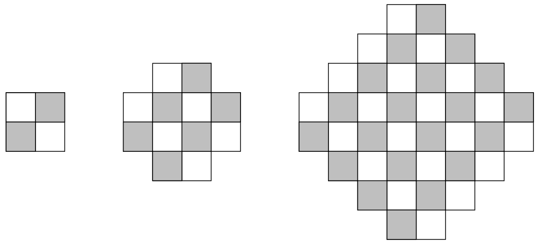

Fig. 1.1. From left to right, the Aztec diamonds of order 1, 2 and 4.

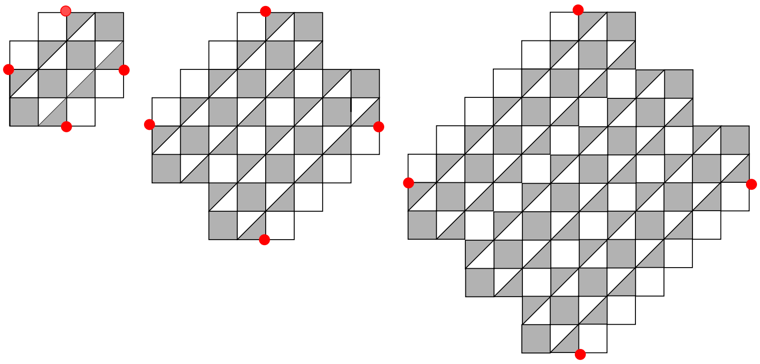

Fig. 1.2. From left to right, the Douglas regions of order 1, 2 and 3.

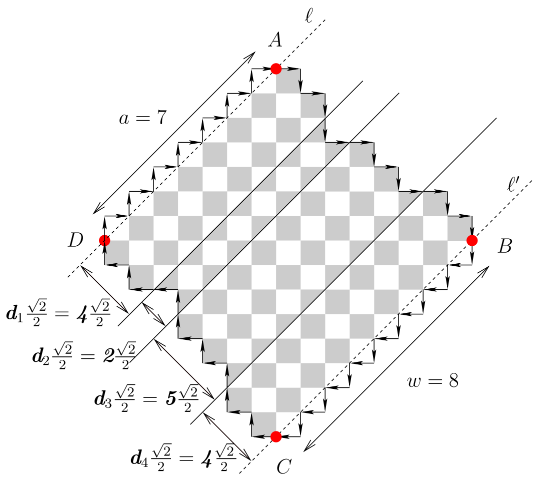

Fig. 1.3. The region $D_7(4, 2, 5, 4)$.

$d_1\frac{\sqrt{2}}{2}, d_2\frac{\sqrt{2}}{2}, \ldots, d_{k-1}\frac{\sqrt{2}}{2}$, for some positive integers d_i (see Fig. 1.3). The above set-up of drawn-in diagonals gives a new lattice whose fundamental regions are unit squares or triangles (halves of a unit square). We notice that the triangles only appear along the drawn-in diagonals.

Given a positive integer a, we define the region $D_a(d_1, \ldots, d_k)$ as follows (see Fig. 1.3 for an example). The southwestern and northeastern boundaries of $D_a(d_1, \ldots, d_k)$ are defined in the next paragraph.

Color the new lattice black and white so that any two fundamental regions sharing an edge have opposite colors, and the fundamental regions passed through by ℓ are white. Starting from a lattice point A on ℓ, we take unit steps south or east so that for each step the fundamental region on the right is black. We meet ℓ' at another lattice point B; and the described path from A to B is the northeastern boundary of our region. Next, we pick the lattice point D on ℓ to the left of A so that the distance between A and D is $a\sqrt{2}$. The southwestern boundary of our region is obtained by reflecting the northeastern one about the perpendicular bisector of segment AD, and reversing the directions of its steps (from south to north, and from east to west). Let C be the intersection of the southwestern boundary and ℓ'.