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Recursive constructions and nonisomorphic minimal nonorientable embeddings of complete graphs

Vladimir P. Korzhik *

National University of Chernivtsi, Chernivtsi, Ukraine
Institute of Applied Problems of Mechanics and Mathematics of National Academy of Science of Ukraine, Lviv, Ukraine

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ABSTRACT

We construct a family of recursive constructions such that for any $i \in \{0, 1, 3, 4, 6, 7, 9, 10\}$ and $j \in \{0, 1, \dots, 11\}$, several arbitrary nonorientable triangular embeddings of every complete graph K_m, $m \equiv i \pmod{12}$, can be incorporated into a minimal nonorientable embedding of $K_{\bar{m}}$, $\bar{m} \equiv j \pmod{12}$. The existence of such recursive constructions implies the following important interdependency of the sets of nonisomorphic minimal nonorientable embeddings of K_n for different residue classes of n modulo 12: if for some $i \in \{0, 1, 3, 4, 6, 7, 9, 10\}$, the number of nonisomorphic nonorientable triangular embeddings of a graph K_m, $m \equiv i \pmod{12}$, is large enough, then for any other $j \in \{0, 1, \dots, 11\}$, the number of nonisomorphic minimal nonorientable embeddings of some graph $K_{\bar{m}}$, $\bar{m} \equiv j \pmod{12}$, is also large enough. As a consequence, using Grannell and Knor's (2013) results for K_n, $n \equiv 1$ or $9 \pmod{12}$, we show that there is a certain positive constant a such that for any $i \in \{0, 1, \dots, 11\}$, there is an infinite set (a linear class) of values of n, where $n \equiv i \pmod{12}$, such that the number of nonisomorphic minimal nonorientable embeddings of K_n is at least $n^{an^2 - o(n^2)}$ as $n \to \infty$.

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1. Introduction

We can differentiate cellular embeddings of graphs in surfaces as labeled objects (in this case we speak about different labeled embeddings, they have different face sets) and as unlabeled objects (in this case we speak about nonisomorphic embeddings). In what follows we consider only cellular embeddings of graphs.

Let K be a graph without loops and multiple edges. An m-gonal face of an embedding of K will be designated as a cyclic sequence $[v_1, v_2, \dots, v_m]$ of vertices obtained by listing the incident vertices when traversing the boundary walk of the face in some chosen direction. The sequences $[v_1, v_2, \dots, v_m]$ and $[v_m, \dots, v_2, v_1]$ designate the same face.

Two embeddings f and f' of a graph K are *different labeled* embeddings if f has and f' does not have a face $[v_1, v_2, \dots, v_m]$. Two embeddings f and f' of K are *isomorphic* if there is an automorphism ψ of K such that if $[v_1, v_2, \dots, v_m]$ is a face of f, then $[\psi(v_1), \psi(v_2), \dots, \psi(v_m)]$ is a face of f'.

The nonorientable genus of a graph is the smallest q such that the graph can be embedded in N_q, the sphere with q crosscaps attached; any such embedding is called a *minimal* nonorientable embedding of the graph. During the proof of the Map Color Theorem for nonorientable surfaces [10] one minimal nonorientable embedding was constructed for every complete graph K_n. Constructing minimal nonorientable embeddings of K_n was carried in a different way for twelve cases: for

* Correspondence to: Bogomoltsa St. 3/5, Chernivtsi, 58001, Ukraine.
E-mail address: korzhikvp@gmail.com.

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every $i \in \{0, 1, \dots, 11\}$, some specific rotation schemes (and later specific current graphs) were used to construct minimal nonorientable embeddings of K_n for all $n \equiv i \pmod{12}$.

An embedding of a graph is *triangular* if every face is 3-gonal. Euler's formula allows a possibility for the complete graph K_n to have a triangular embedding in a nonorientable surface if $n \equiv 0$ or $1 \pmod 3$ only, and we have that only for $n \equiv 0$ or $1 \pmod 3$ the minimal nonorientable embeddings of K_n are triangular. We will use the acronym NTE for a nonorientable triangular embedding.

In this paper we consider the natural question on the number of nonisomorphic minimal nonorientable embeddings of complete graphs.

It is known (see [2]) that the number of nonisomorphic triangular embeddings of the complete graph K_n cannot exceed $n^{n^2/3}$.

Using recursive constructions and a cut-and-paste technique it was shown [1,4] that there are at least $2^{n^2/54-o(n^2)}$ nonisomorphic face 2-colorable NTE's of K_n for some families of n such that $n \equiv 1$ or $3 \pmod 6$). This approach having to do with face 2-colorable triangular embeddings does not work in the case of complete graphs of even order and does not work in the case of constructing nontriangular minimal nonorientable embeddings of complete graphs.

Index one current graphs were applied [9] to show that there are constants $M, c > 0$, $b \geq 1/12$ such that for every $n \geq M$, there are at least $c2^{bn}$ nonisomorphic minimal (triangular for $n \not\equiv 2 \pmod 3$, and nontriangular for $n \equiv 2 \pmod 3$)) nonorientable embeddings of K_n. Up to the present paper, this result was the only known in the literature result on the number of nonisomorphic nontriangular minimal nonorientable embeddings of complete graphs.

Recently, using face 2-colorable triangular embeddings of complete tripartite graphs it was shown [2] (see also [3,5–7]) that, for a certain positive constant a and for an infinite number of values of n, where $n \equiv 1$ or $9 \pmod{12}$, the number of nonisomorphic NTE's of K_n is at least $n^{an^2-o(n^2)}$. At first [2], these results were obtained for rather sparse sets of values of n (by "sparse"we mean that the number of suitable values not exceeding m is of order $\log_2 m$). More recently [7] lower bounds of the form n^{an^2} were established for linear classes of values of n (by a *linear class* of values of n we mean an infinite set $\{a + bt : t = 1, 2, \dots\}$ of values of n, where a and b are integer constants). This approach having to do with face 2-colorable triangular embeddings does not work in the case of complete graphs of even order and does not work in the case of constructing nontriangular minimal nonorientable embeddings of complete graphs.

In the present paper we construct a family of recursive constructions such that for any $i \in \{0, 1, 3, 4, 6, 7, 9, 10\}$ and $j \in \{0, 1, \dots, 11\}$, several arbitrary NTE's of every complete graph K_m, $m \equiv i \pmod{12}$, can be incorporated into a minimal nonorientable embedding of $K_{\bar{m}}$, $\bar{m} \equiv j \pmod{12}$. The existence of such recursive constructions implies the following important interdependency of the sets of nonisomorphic minimal nonorientable embeddings of K_n for different residue classes of n modulo 12: if for some $i \in \{0, 1, 3, 4, 6, 7, 9, 10\}$, the number of nonisomorphic nonorientable triangular embeddings of a graph K_m, $m \equiv i \pmod{12}$, is large enough, then for any other $j \in \{0, 1, \dots, 11\}$, the number of nonisomorphic minimal nonorientable embeddings of some graph $K_{\bar{m}}$, $\bar{m} \equiv j \pmod{12}$, is also large enough. As a consequence, using Grannell and Knor's (2013) results for K_n, $n \equiv 1$ or $9 \pmod{12}$, we show that there is a certain positive constant a such that for any $i \in \{0, 1, \dots, 11\}$, there is an infinite set (a linear class) of values of n, where $n \equiv i \pmod{12}$, such that the number of nonisomorphic minimal nonorientable embeddings of K_n is at least $n^{an^2-o(n^2)}$ as $n \to \infty$. We also obtain (Theorem 2) lower bounds of the form $2^{bn^2-o(n^2)}$ (as $n \to \infty$) on the number of nonisomorphic minimal nonorientable embeddings of complete graphs K_n for some classes of even n and of complete graphs K_n for some classes of odd n, where $n \equiv 2 \pmod 3$.

The paper is organized as follows. In Section 2 we prove the main results of the paper. The proofs are based on the existence of recursive constructions described in Section 4. To obtain the recursive constructions we use index 2, 3, and 4 current graphs. To facilitate construction checking in Section 4, some material about index 2, 3, and 4 current graphs is given in Section 3.

2. Recursive constructions and nonisomorphic minimal embeddings

By a *one-hamiltonian embedding* of a complete graph K_m we mean an embedding such that one face is m-gonal and bounded by a hamiltonian cycle of the graph, and all other faces are triangular. Given a triangular embedding of K_n, if we delete a vertex of the embedded graph, we obtain a one-hamiltonian embedding of K_{n-1} which is said to be induced by the triangular embedding of K_n. Clearly, if two one-hamiltonian embeddings f and f' of K_{n-1} induced, respectively, by triangular embeddings φ_1 and φ_2 of K_n are isomorphic, then the embeddings φ_1 and φ_2 are isomorphic as well. Hence we have the following.

Claim 1. *If for every of M nonisomorphic NTE's of K_n we take a one-hamiltonian embedding of K_{n-1} induced by the triangular embedding, we obtain M nonisomorphic nonorientable one-hamiltonian embeddings of K_{n-1}.*

For $k(n-1) \leq m$, denote by $G(n|k|m)$ the m-vertex graph which is obtained if we take vertex-disjoint $k(n-1)$-gonal cycles $C_1, C_2, \dots, C_k$ (they are called the *special cycles* of G) and an $(m-k(n-1))$-vertex set and then join by an edge any pair of vertices not belonging to the same special cycle. For $m \not\equiv 2 \pmod 3$, by an embedding $R(n|k|m)$ we mean an embedding of the graph $G(n|k|m)$ such that there are $k(n-1)$-gonal faces with boundaries $C_1, C_2, \dots, C_k$, respectively, and all other

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