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Domination game on forests

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ABSTRACT

In the domination game studied here, Dominator and Staller alternately choose a vertex of a graph G and take it into a set D. The number of vertices dominated by the set D must increase with each move and the game ends when D becomes a dominating set of G. Dominator aims to minimize while Staller aims to maximize the number of turns (or equivalently, the size of the dominating set D obtained at the end). Assuming that Dominator starts and both players play optimally, the number of turns is the *game domination number* $\gamma_g(G)$ of G.

Kinnersley, West, and Zamani proved that $\gamma_g(G) \le 7n/11$ for every isolate-free n-vertex forest G, and they conjectured that the sharp upper bound is only $3n/5$. Here, we prove the 3/5-conjecture for forests in which no two leaves are at distance 4 apart. Further, we establish an upper bound $\gamma_g(G) \le 5n/8$ for every n-vertex isolate-free forest G.

1. Introduction

1.1. Domination game

The domination game considered here was introduced in 2010 by Brešar, Klavžar and Rall [3], where the original idea is attributed to Henning in 2003. For this *domination game*, a graph G is given and two players, called *Dominator* and *Staller*, take turns choosing a vertex and taking it into a set D. Each vertex chosen dominates itself and its neighbors. It is required that the set of vertices dominated by D must be enlarged with each move. The game ends when no more legal moves can be made; that is, when D becomes a dominating set of G. The goal of Dominator is to minimize while that of Staller is to maximize the length of the game. Equivalently, Dominator wants a small dominating set D, while Staller wants D to be as large as possible. The *game domination number* $\gamma_g(G)$ of G is the number of turns in the game (it equals the size of the dominating set D obtained at the end) when Dominator starts the game and both players play optimally. Analogously, the *Staller-start game domination number* $\gamma_g'(G)$ is the number of turns when Staller begins and the players play optimally.

1.2. Standard definitions

The *open neighborhood* $N_G(v)$ of a vertex v in a graph G is defined by $N_G(v) = \{u\colon uv \in E(G)\}$, while its *closed neighborhood* $N_G[v]$ is defined by $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $d_G(v)$ of v is just $|N_G(v)|$. Each vertex dominates itself and its neighbors. The *closed neighborhood* $N_G[S]$ of a set $S \subseteq V(G)$ is defined by $N_G[S] = \bigcup_{v\in S} N_G[v]$, and S *dominates* its closed neighborhood. Instead of $N_G(v)$, $N_G[v]$, $N_G[S]$, and $d_G(v)$ we will write $N(v)$, $N[v]$, $N[S]$ and $d(v)$, respectively, if G is clear from the context.

A vertex set $D \subseteq V(G)$ is a *dominating set* of G if $N_G(D) = V(G)$. The minimum size of a dominating set D is the *domination number* $\gamma(G)$ of G. It is easy to prove that $\gamma(G) \le \gamma_g(G) \le 2\gamma(G) - 1$ and $\gamma(G) \le \gamma_g'(G) \le 2\gamma(G)$ hold.

In a tree, as usual, a leaf is a vertex of degree 1, while a vertex having a leaf neighbor is a *stem*.

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1.3. Results on the domination game

The earlier papers discuss several aspects of the domination game, for example, connections between $\gamma_g(G)$ and $\gamma_g'(G)$ [3,7,8], the game domination number of Cartesian products [3] moreover the difference between $\gamma_g(G)$ and $\gamma_g(H)$ when H is a spanning subgraph of G [4]. Also, [1] discusses the possible changes of the game domination number when a vertex or an edge is deleted.

From our point of view, the following "3/5-conjecture" and the related results are the most important ones.

Conjecture 1 (*Kinnersley, West, and Zamani [7]*)**.** *If G is an isolate-free forest of order n, then*

$$\gamma_g(G) \le \frac{3n}{5} \quad \text{and} \quad \gamma_g'(G) \le \frac{3n+2}{5}.$$

Conjecture 1 was proved in [7] for graphs whose components are all caterpillars.[1] Additionally, in [2] computer search was used to determine all trees up to 20 vertices attaining this bound, and infinitely many trees G such that $\gamma_g(G) = 3n/5$ were constructed. Hence, the bound $3n/5$ (if true) is sharp.

One of our contributions is the proof of Conjecture 1 for the class of forests in which no two leaves are connected by a path of length 4. For this class of forests our upper bound $(3n+1)/5$ on γ_g' is slightly better than the bound conjectured in [7] for forests in general.

Theorem 1. *If G is an isolate-free forest of order n in which no two leaves have distance 4, then*

$$\gamma_g(G) \le \frac{3n}{5} \quad \text{and} \quad \gamma_g'(G) \le \frac{3n+1}{5}.$$

Our proof, presented in Section 3, is based on a value-assignment to the vertices, where the value of a vertex v depends on the current status of v in the game. We then describe a greedy-like strategy for Dominator which ensures that the game ends within $3n/5$ turns. We introduced this approach in the conference paper [5], where also Theorem 1 was stated without a completely detailed proof. The strategy described there is fine-tuned here, and the proof is extended by a more detailed analysis to obtain a further result. This new general upper bound $5n/8$ concerns all isolate-free forests and improves the earlier bound $\gamma_g(G) \le 7n/11$, which was recently proved by Kinnersley, West, and Zamani [7].

Theorem 2. *If G is an isolate-free forest of order n, then*

$$\gamma_g(G) \le \frac{5n}{8} \quad \text{and} \quad \gamma_g'(G) \le \frac{5n+2}{8}.$$

The paper is organized as follows. In Section 2, the basic value-assignment is introduced and some general lemmas are obtained. In Section 3, we describe the strategy and analyze the structure of the residual graph at some crucial points. In the last subsection of this part, we verify Theorems 1 and 2 based on the previous lemmas. In Section 4, we make some concluding notes.

2. Preliminaries

At any moment of the game we have three different types of vertices. We assign to them different colors and different values. The letter D always denotes the set of vertices selected by the players up to the current time. A vertex v is *dominated* if $v \in N[D]$; otherwise v is *undominated*.

- A vertex is *white* and its value is 3 if it is undominated.
- A vertex is *blue* and its value is 2 if it is dominated but has at least one undominated neighbor.
- A vertex is *red* and its value is 0 if it and all of its neighbors are dominated.

Clearly, selecting a red vertex would not enlarge the set of dominated vertices, hence this choice is not legal in the game. Also, selecting any vertex does not change the status of a red vertex. Hence, red vertices can be ignored in the continuation of the game. On the other hand, blue vertices can be chosen later by either player, since they have white neighbors, but edges connecting two blue vertices can be deleted. Therefore, at any given point in the game, the graph G will be understood not to have red vertices or edges joining two blue vertices. This graph G is the *residual graph* introduced in [7]. Due to our definition, in a residual graph each blue vertex has only white neighbors and has at least one neighbor. For white vertices, none of their neighbors and none of the edges incident with at least one white vertex were deleted. This implies the following statements.

[1] A caterpillar is a tree whose non-leaf vertices induce a path.