Discrete Mathematics 338 (2015) 2234–2241

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Low edges in 3-polytopes

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ARTICLE INFO

Article history:
Received 19 June 2014
Received in revised form 18 May 2015
Accepted 19 May 2015
Available online 20 June 2015

Keywords:
Plane map
Plane graph
3-polytope
Structural property
Height of edge

ABSTRACT

The height $h(e)$ of an edge e in a 3-polytope is the maximum degree of the two vertices and two faces incident with e. In 1940, Lebesgue proved that every 3-polytope without so called pyramidal edges has an edge e with $h(e) \leq 11$. In 1995, this upper bound was improved to 10 by Avgustinovich and Borodin. Recently, we improved it to 9 and constructed a 3-polytope without pyramidal edges satisfying $h(e) \geq 8$ for each e.

The purpose of this paper is to prove that every 3-polytope without pyramidal edges has an edge e with $h(e) \leq 8$.

In different terms, this means that every plane quadrangulation without a face incident with three vertices of degree 3 has a face incident with a vertex of degree at most 8, which is tight.

1. Introduction

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [31], 3-polytopes are in 1–1 correspondence with 3-connected planar graphs.

The *degree* $d(x)$ of a vertex or face x in a 3-polytope M is the number of incident edges. A *k-vertex* and *k-face* is one of degree k, a k^+*-vertex* has degree at least k, and so on. *Elements* of a 3-polytope are its vertices and faces.

The *height* $h(e)$ of an edge e in M is the maximum of the degrees of two vertices and two faces incident with e.

An edge is *pyramidal* if it is incident with at least three elements of degree 3 (in fact, edges incident with four elements of degree 3 exist only in the tetrahedron). Note that each edge of the n-pyramid has height n. Thus if pyramidal edges are allowed in a 3-polytope, it can happen that all its edges are of unbounded height.

In 1940, Lebesgue [23] proved that every 3-polytope without pyramidal edges has an edge of height at most 11. In 1995, this bound was lowered to 10 in Avgustinovich–Borodin [1].

A well-known lower bound 7 on the height of edges is obtained by capping every 4-face of the Archimedean (3, 4, 4, 4)-solid (in which every vertex is incident with a 3-face and three 4-faces).

Recently, we improved [9] the upper bound to 9 and constructed a 3-polytope satisfying $h(e) \geq 8$ for each e.

The purpose of this paper is to prove that 8 is attained from above.

Theorem 1. *Every* 3*-polytope without pyramidal edges contains an edge of height at most* 8*, which is tight.*

In addition to the pyramid, the necessity of forbidding pyramidal edges in Theorem 1 is confirmed by the following construction, in which every edge is arbitrarily high and which contains non-pyramidal edges.

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http://dx.doi.org/10.1016/j.disc.2015.05.018

We take a double $2n$-pyramid with $2n$-vertices x, z and a cycle $y_1 \dots y_{2n}$ of 4-vertices. Every edge $y_k y_{k+1}$ with $1 \le k \le 2n$ (addition modulo $2n$) is replaced by a path $y_k y_{k,1} \dots y_{k,n-3} y_{k+1}$, where all new vertices are of degree 2. Finally, whenever $1 \le k \le 2n$, we join all 2-vertices $y_{k,1}, \dots, y_{k,n-3}$ to x if k is even or to z otherwise. It remains to note that every edge has height at least n and every edge incident with a 4-vertex is non-pyramidal.

Theorem 1 can easily be translated into the language of quadrangulations as follows.

Theorem 2. *Every quadrangulated 3-polytope without faces incident with three 3-vertices has a face incident with 8^--vertices only, which is tight.*

To deduce Theorem 2 from Theorem 1, we take a quadrangulation Q (which is a bipartite graph), color its vertices with colors V and F, join the vertices colored V in each face by an edge, and delete all edges of Q. Clearly, an edge of height at most 8 in the polytope obtained produces a face of height at most 8 in Q.

We recall some results on the structure of 5^--faces in 3-polytopes. By Δ and δ denote the maximum and minimum vertex degree of M, respectively. The *weight* of a face in M is the degree sum of its boundary vertices, and $w(M)$, or simply w, denotes the minimum weight of 5^--faces in M.

We say that f is a *face of type* $(k_1, k_2, \dots)$ or simply a $(k_1, k_2, \dots)$-*face* if the set of degrees of the vertices incident with f is majorized by the vector $(k_1, k_2, \dots)$.

Back in 1940, Lebesgue [23] described 5^--faces in normal plane maps.

Theorem 3 (*Lebesgue [23]*)**.** *Every normal plane map has a 5^--face of one of the following types:*
$(3, 6, \infty)$, $(3, 7, 41)$, $(3, 8, 23)$, $(3, 9, 17)$, $(3, 10, 14)$, $(3, 11, 13)$,
$(4, 4, \infty)$, $(4, 5, 19)$, $(4, 6, 11)$, $(4, 7, 9)$, $(5, 5, 9)$, $(5, 6, 7)$,
$(3, 3, 3, \infty)$, $(3, 3, 4, 11)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$, $(3, 3, 3, 3, 5)$.

The classical Theorem 3, along with other ideas in Lebesgue [23], has a lot of applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [7,28,30]).

Some parameters of Lebesgue's Theorem were improved for narrow classes of plane graphs. In 1963, Kotzig [21] proved that every plane triangulation with $\delta = 5$ satisfies $w \le 18$ and conjectured that $w \le 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 4 (*Borodin [2]*)**.** *Every normal plane map with $\delta = 5$ has a $(5, 5, 7)$-face or a $(5, 6, 6)$-face, where all parameters are tight.*

Theorem 4 also confirmed a conjecture of Grünbaum [16] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [29]).

We note that w is unbounded in the set of all 3-polytopes with $(4, 4, \infty)$-faces, as follows from the n-pyramid, double n-pyramid, and a related construction in which every 3-face is incident with a 3-vertex, 4-vertex, and n-vertex. The same is true concerning $(3, 3, 3, \infty)$-faces: take the double $2n$-pyramid and delete all even upper spokes and all odd lower ones to obtain a quadrangulation having only $(3, 3, 3, n)$-faces.

For plane triangulations without 4-vertices, Kotzig [22] proved $w \le 39$, and Borodin [4], confirming Kotzig's conjecture in [22], proved $w \le 29$, which is best possible due to the dual of the twice-truncated dodecahedron. Borodin [5] further shows that each triangulated 3-polytope without $(4, 4, \infty)$-faces satisfies $w \le 29$, and that for triangulations without 4-vertices adjacent to each other there is a sharp bound $w \le 37$.

For an arbitrary 3-polytope, Theorem 3 yields $w \le \max\{51, \Delta+9\}$. Horňák and Jendrol' [17] strengthened this as follows: if there are neither $(4, 4, \infty)$-faces nor $(3, 3, 3, \infty)$-faces, then $w \le 47$. Borodin and Woodall [12] proved that forbidding $(3, 3, 3, \infty)$-faces implies $w \le \max\{29, \Delta + 8\}$.

For quadrangulated 3-polytopes, Avgustinovich and Borodin [1] improved the description of 4-faces implied by Lebesgue's Theorem as follows: $(3, 3, 3, \infty)$, $(3, 3, 4, 10)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$.

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in a recent survey by Jendrol' and Voss [19], and also in [3,11,13–15,18,20,24,26,25,27,32].

In 2002, Borodin [6] strengthened Lebesgue's Theorem 3 as follows (the entries marked by an asterisk are proved in [6] to be best possible).

Theorem 5 (*Borodin [6]*)**.** *Every normal plane map has a 5^--face of one of the following types:*
$(3, 6, \infty^*)$, $(3, 8^*, 22)$, $(3, 9^*, 15)$, $(3, 10^*, 13)$, $(3, 11^*, 12)$,
$(4, 4, \infty^*)$, $(4, 5^*, 17)$, $(4, 6^*, 11)$, $(4, 7^*, 8)$, $(5, 5^*, 8)$, $(5, 6, 6^*)$,
$(3, 3, 3, \infty^*)$, $(3, 3, 4^*, 11)$, $(3, 3, 5^*, 7)$, $(3, 4, 4, 5^*)$, $(3, 3, 3, 3, 5^*)$.

Recently, precise descriptions of the structure of faces were obtained for 3-polytopes with $\delta \ge 4$ and for triangulated 3-polytopes.