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Perfect matchings avoiding prescribed edges in a star-free graph

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ABSTRACT

Aldred and Plummer (1999) have proved that every m-connected $K_{1,m-k+2}$-free graph of even order contains a perfect matching which avoids k prescribed edges. They have also proved that the result is best possible in the range $1 \leq k \leq \frac{1}{2}(m+1)$. In this paper, we show that if $\frac{1}{2}(m+2) \leq k \leq m-1$, their result is not best possible. We prove that if $m \geq 4$ and $\frac{1}{2}(m+2) \leq k \leq m-1$, every $K_{1,\lceil \frac{2m-k+4}{3} \rceil}$-free graph of even order contains a perfect matching which avoids k prescribed edges. While this is a best possible result in terms of the order of a forbidden star, if $2m-k+4 \equiv 0 \pmod 3$, we also prove that only finitely many sharpness examples exist.

1. Introduction

For a connected graph H, a graph G is said to be H-free if G does not contain an induced subgraph isomorphic to H. In particular, a $K_{1,3}$-free graph is called a claw-free graph, and given an integer $n \geq 3$, a $K_{1,n}$-free graph is called a star-free graph. Sumner [9] studied the existence of a perfect matching in a star-free graph, and proved the following theorem. The first part was also proved by Las Vergnas [5].

Theorem A (*[9,5]*).

(1) *A connected claw-free graph of even order has a perfect matching.*
(2) *For every integer m with $m \geq 2$, an m-connected $K_{1,m+1}$-free graph of even order has a perfect matching.*

This theorem is best possible in the following senses. There are infinitely many connected $K_{1,4}$-free graphs of even order without a perfect matching, and for $m \geq 2$, infinitely many m-connected $K_{1,m+2}$-free graphs of even order without a perfect matching.

Plummer and Saito [7] investigated a sort of converse of Theorem A. They studied whether it is possible to guarantee the existence of a perfect matching by forbidding a connected graph other than a star, and gave the following negative answer.

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Theorem B ([7]). *Let H be a connected graph of order at least three, and let m be an integer with $m \geq 2$.*

(1) *If there exists a positive integer n_0 such that every connected H-free graph of even order at least n_0 has a perfect matching, then either $H = K_{1,3}$ or $H = K_{1,2}$.*
(2) *If there exists a positive integer n_0 such that every m-connected H-free graph of even order at least n_0 has a perfect matching, then $H = K_{1,l}$ for some l with $2 \leq l \leq m + 1$.*

According to Theorem B, as long as we forbid one graph to guarantee the existence of a perfect matching, Theorem A gives a complete solution, and the situation does not change even if we allow a finite number of exceptions.

Plummer [6] introduced the notion of a k-extendable graph. Let k be a nonnegative integer and let G be a connected graph of even order at least $2k + 2$ with a perfect matching. Then G is said to be *k-extendable* if every set of k independent edges lies in some perfect matching of G. This notion was later extended to the property $E(m, n)$ by Porteous and Aldred [8]. Let m and n be nonnegative integers and let G be a connected graph of even order at least $2(m + n + 1)$. Then G is said to be $E(m, n)$ if for every pair of disjoint sets of independent edges M and N of order m and n, respectively, there exists a perfect matching L with $M \subset L$ and $N \cap L = \emptyset$.

Chen [3] studied the extendability of star-free graphs, and proved the following theorem.

Theorem C ([3]). *Let m and k be integers with $1 \leq k \leq \frac{1}{2}m$. Then every m-connected $K_{1,m-2k+2}$-free graph of even order at least $2k + 2$ is k-extendable.*

This result was extended by Aldred and Plummer [1].

Theorem D ([1]). *Let m, k and l be integers with $1 \leq 2l + k \leq m$. Then every m-connected $K_{1,m-2l-k+2}$-free graph of even order at least $2(k + l + 1)$ is $E(l, k)$.*

However, the proof of Theorem D in [1] does not use the fact that the k deleted edges form a matching. Therefore, it actually proves the following stronger statement.

Theorem E ([1]). *Let m, k and l be integers with $1 \leq 2l + k \leq m$. Let G be an m-connected $K_{1,m-2l-k+2}$-free graph of even order at least $2(k + l + 1)$. Then for every $F \subset E(G)$ with $|F| = k$, $G - F$ is l-extendable.*

In order to highlight the theme of this paper, we set $l = 0$.

Theorem F. *Let k and m be integers with $1 \leq k \leq m$. Let G be an m-connected $K_{1,m-k+2}$-free graph of even order at least $2(k + 1)$. Then for every $F \subset E(G)$ with $|F| \leq k$, $G - F$ has a perfect matching.*

Theorem F is best possible if $0 \leq k \leq \frac{1}{2}(m + 1)$. For a positive integer r, let $G_r = K_m + (kK_2 \cup (m - 2k + 1)K_1 \cup K_{2r-1})$, where for graphs G and H, we denote by $G + H$ and $G \cup H$ the join and the union of G and H, respectively, and for a positive integer n, we denote by nG the union of n copies of G. Then G_r is an m-connected $K_{1,m-k+3}$-free graph of even order, but G_r does not have a perfect matching which avoids the edges in kK_2.

Though the above example shows the sharpness of Theorem F, it only exists in the range $0 \leq k \leq \frac{1}{2}(m + 1)$. The purpose of this paper is to prove that in the range $\frac{1}{2}(m + 2) \leq k \leq m - 1$, Theorem F is no longer best possible. For this range of k, we prove the following theorem, which is our main result.

Theorem 1. *Let k and m be integers with $m \geq 4$ and $\frac{1}{2}(m + 2) \leq k \leq m - 1$. Let G be an m-connected $K_{1,\lceil \frac{2m-k+4}{3} \rceil}$-free graph of even order. Then for every $F \subset E(G)$ with $|F| = k$, $G - F$ has a perfect matching.*

If $k > \frac{1}{2}(m + 2)$, then $\lceil \frac{2m-k+4}{3} \rceil > m - k + 2$. Therefore, Theorem 1 forbids a larger star than that in Theorem D and still guarantees the existence of a perfect matching avoiding k prescribed edges.

In the next section, we introduce several invariants of a multigraph and study their properties. Using them, we prove the main theorem in Section 3. In Section 4, we discuss the sharpness of the result. And in Section 5, we give concluding remarks.

As far as the statement of Theorem 1 is concerned, multiple edges have little meaning. However, in the proof of the theorem, we deal with an auxiliary graph which may have multiple edges. Therefore, we strictly distinguish the term "graph" and "multigraph" in this paper. When we say a "multigraph", we refer to a situation in which multiple edges are allowed. On the other hand, when we say a "graph" or a "simple graph", we refer to a graph without multiple edges. Loops are not allowed throughout this paper. For a multigraph H, the *underlying graph* of H is the graph obtained from H by replacing every multiple edge with a single edge.

For standard graph-theoretic notation and terminology, we refer the reader to [2]. Let G be a multigraph and let $x \in V(G)$. We denote by $N_G(x)$ and $\deg_G(x)$ the neighborhood and the degree of x in G, respectively. Let X and Y be sets of vertices in G with $X \cap Y = \emptyset$. Then we define $N_G(X)$ by $N_G(X) = \bigcup_{x \in X} N_G(x)$. Also we denote by $E_G(X, Y)$ the set of edges in G with one endvertex in X and the other in Y, and we define $e_G(X, Y)$ by $e_G(X, Y) = |E_G(X, Y)|$. Furthermore, $G[X]$ denotes the subgraph of G induced by X. We denote by $\alpha(G)$, $\Delta(G)$, $w(G)$ and $c_o(G)$ the independence number, the maximum degree, the number of components and the number of odd components, respectively, of G. For $F \subset E(G)$, let $V(F)$ denote the set of vertices that are endvertices of the edges in F. Let $|G|$ denote the order of G.

We denote by P_n and C_n the path and the cycle of order n, respectively. While a path is always a simple graph, we shall consider C_2, a cycle of order two, in some parts of the proof, which is a multigraph consisting of two vertices joined by two edges.