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Brooks type results for conflict-free colorings and $\{a, b\}$-factors in graphs

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ABSTRACT

A vertex-coloring of a hypergraph is *conflict-free*, if each edge contains a vertex whose color is not repeated on any other vertex of that edge. Let $f(r, \Delta)$ be the smallest integer k such that each r-uniform hypergraph of maximum vertex degree Δ has a conflict-free coloring with at most k colors. As shown by Pach and Tardos, similarly to a classical Brooks' type theorem for hypergraphs, $f(r, \Delta) \leq \Delta + 1$. Compared to Brooks' theorem, according to which there is only a couple of graphs/hypergraphs that attain the $\Delta + 1$ bound, we show that there are several infinite classes of uniform hypergraphs for which the upper bound is attained. We provide bounds on $f(r, \Delta)$ in terms of Δ for large Δ and establish the connection between conflict-free colorings and so-called $\{t, r - t\}$-factors in r-regular graphs. Here, a $\{t, r - t\}$-factor is a factor in which each degree is either t or $r - t$. Among others, we disprove a conjecture of Akbari and Kano (2014) stating that there is a $\{t, r - t\}$-factor in every r-regular graph for odd r and any odd $t < \frac{r}{3}$.
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1. Introduction

Given a vertex coloring of a hypergraph, we call a vertex contained in an edge E *uniquely colored in* E, if its color is assigned to no other vertex in E. If every edge of a hypergraph contains a uniquely colored vertex, then the coloring is called *conflict-free*. The *conflict-free chromatic number* $\chi_{\text{cf}} = \chi_{\text{cf}}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the minimum number of colors used in a conflict-free coloring of $\mathcal{H}$.

Conflict-free colorings are closely related to proper colorings in which each hyperedge is not monochromatic, i.e., contains at least two vertices of distinct colors. The chromatic number $\chi(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the smallest number of colors in a proper coloring of $\mathcal{H}$. The following Brooks' type theorem for hypergraphs proved by Kostochka, Stiebitz and Wirth [14] generalizes Brooks' theorem for graphs. The maximum degree $\Delta(\mathcal{H})$ of a hypergraph is the largest number of hyperedges containing a common vertex. A hypergraph is connected if for any two vertices u, v there is a sequence of vertices starting with u and ending with v such that any two consecutive vertices in this sequence belong to a hyperedge.

Theorem 1 ([14]). *Let $\mathcal{H} = (V, \mathcal{E})$ be a connected hypergraph with $|E| \geq 2$ for each edge E. If $|\mathcal{E}| > 1$ and $\mathcal{H}$ is neither an ordinary odd cycle nor an ordinary complete graph, then $\chi(\mathcal{H}) \leq \Delta(\mathcal{H})$. Otherwise $\chi_{\text{cf}}(\mathcal{H}) = \chi(\mathcal{H}) = \Delta(\mathcal{H}) + 1$.*

Pach and Tardos provided the same upper bound on χ_{cf}:

Theorem 2 ([18]). *For each hypergraph $\mathcal{H}$, $\chi_{\text{cf}}(\mathcal{H}) \leq \Delta(\mathcal{H}) + 1$.*

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In this paper, we find infinite classes of hypergraphs for which the upper bound $\Delta + 1$ on the conflict-free chromatic number is attained and provide general bounds in terms of the maximum degree for uniform hypergraphs. In addition, we provide a connection between conflict-free colorings of hypergraphs and so-called $\{t, r - t\}$-factors in r-regular graphs. Here, a $\{t, r - t\}$-factor is a spanning subgraph with every vertex of degree t or $r - t$.

Next we list the main results of the paper. A hypergraph is r-uniform if each hyperedge contains exactly r vertices. The main parameter we introduce is

$$f(r, \Delta) = \max\{\chi_{\mathrm{cf}}(\mathcal{H}) : \ \mathcal{H} \text{ is } r\text{-uniform}, \ \ \Delta(\mathcal{H}) = \Delta\}.$$

In Theorem 3 we provide bounds on $f(r, \Delta)$ when r is fixed and Δ is large. Theorem 4 provides Brooks type results.

Theorem 3. *For any integer $r \geq 3$, there are constants c, c', and d depending on r such that for any $\Delta > d$*

$$c\frac{\Delta^{1/\lceil \frac{r}{2} \rceil}}{\ln(\Delta)} \leq f(r, \Delta) \leq c'\Delta^{1/\lceil \frac{r}{2} \rceil}.$$

Theorem 4. *Let r, Δ be positive integers.*
(1) *If $r = 2$ or $(\Delta = 2$ and $r \notin \{3, 5\})$, then $f(r, \Delta) = \Delta + 1$.*
(2) *There are constants c, c_0 such that if $r \geq c$ and $\Delta \geq c_0 \ln r$ then $f(r, \Delta) \leq \Delta$.*
(3) *If $\Delta \geq 3$, then $f(4, \Delta) \leq \Delta$.*

Part of the above theorem is a corollary of an independent result:

Theorem 5. *Let t be an odd positive integer. If r is even or $r \geq (t + 1)(t + 2)$, then there is an r-regular graph that has no $\{t, r - t\}$-factor.*

This disproves the following conjecture.

Conjecture 6 (*Akbari and Kano [1]*)**.** *Let r be an odd integer and a, b denote positive integers such that $a + b = r$ and $a < b$. Then each r-regular graph has an $\{a, b\}$-factor.*

Note that $\chi_{\mathrm{cf}}(\mathcal{H}) = \chi(\mathcal{H})$ if each edge of $\mathcal{H}$ has size 2 or 3. Hence Brooks' theorem and the theorem by Kostochka et al. [14] give a complete characterization of 2- and 3-uniform hypergraphs $\mathcal{H}$ for which $\chi_{\mathrm{cf}}(\mathcal{H}) = \Delta(\mathcal{H}) + 1$. Here, we can also characterize the case of 4-uniform hypergraphs.

Theorem 7. *If $\mathcal{H}$ is a 4-uniform hypergraph on m edges, then $\chi_{\mathrm{cf}}(\mathcal{H}) = \Delta(\mathcal{H}) + 1$ iff $\Delta(\mathcal{H}) = 1$ or $(\Delta(\mathcal{H}) = 2$ and m is odd).*

We provide some history of the problem, background, and basic lemmas in Section 2. An important construction is given in Section 3. Finally, the theorems are proved in Sections 4–6.

2. Background and previous results

Conflict-free colorings were introduced by Even, Lotker, Ron and Smorodinsky [9,21] when considering a frequency assignment problem in wireless networks. A set of base stations in the plane defines the vertices of the hypergraph. A hyperedge is formed by every subset of base stations which is simultaneously reached by a mobile agent from some point in the plane. In order to avoid interferences in communication there should be at least one base station reachable whose frequency is unique among all stations in the range. This corresponds to a conflict-free coloring of the underlying (geometric) hypergraph, where colors correspond to frequencies. Since frequencies are expensive due to limited bandwidth, as few frequencies (respectively colors) as possible shall be used. A survey by Smorodinsky [23] summarizes several results where the hypergraph under consideration is induced by some geometric setting, for example by discs [9,19] or rectangles in the plane [22] and pseudodiscs [11].

Besides these geometrically induced hypergraphs other classes of hypergraphs were considered. Pach and Tardos [18] considered the so called *conflict-free chromatic parameter* of a graph G defined as the conflict-free chromatic number of the hypergraph on the same vertex set $V(G)$ and hyperedges consisting of vertex neighborhoods in the graph. Initially this and lots of related questions were studied by Cheilaris [6]. Recently Glebov, Szabó and Tardos [10] showed upper and lower bounds on this parameter for random graphs in terms of the domination number. In addition, conflict-free colorings of hypergraphs whose hyperedges correspond to simple paths of a given graph were considered [7,8]. A variant of conflict-free colorings using integral colors and each hyperedge containing a vertex whose color is larger than the color of every other vertex in this edge is called a Unique-Maximum coloring or a vertex ranking, see [12,4,15].

Research on general hypergraphs was started by Pach and Tardos [18]. They gave several upper bounds on χ_{cf} in terms of different parameters. The degree of an edge E in a hypergraph $\mathcal{H}$ is the number of edges intersecting E, and the *maximum edge-degree* $D(\mathcal{H})$ is the maximum of the degrees of edges in $\mathcal{H}$. For example Pach and Tardos prove $\chi_{\mathrm{cf}} \in O(t \cdot D^{\frac{1}{t}} \cdot \ln(D))$ if every edge has size at least $2t - 1$. More bounds are due to Kostochka, Kumbhat and Łuczak [13].

Theorem 8 (*[13]*)**.** *If D is the maximum edge-degree of an r-uniform hypergraph $\mathcal{H}$, D is sufficiently large and $D \leq 2^{r/2}$, then $\chi_{\mathrm{cf}}(\mathcal{H}) \leq 120 \ln D$.*

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