Discrete Mathematics 338 (2015) 2348–2355

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Strong edge-coloring for jellyfish graphs

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ARTICLE INFO

Article history:
Received 13 March 2014
Received in revised form 18 April 2015
Accepted 28 April 2015
Available online 23 June 2015

Keywords:
Strong edge-coloring
Strong chromatic index
Cycle
Cactus
Block

ABSTRACT

A strong edge-coloring of a graph is a function that assigns to each edge a color such that two edges within distance two apart receive different colors. The *strong chromatic index* of a graph is the minimum number of colors used in a strong edge-coloring. This paper determines strong chromatic indices of cacti, which are graphs whose blocks are cycles or complete graphs of two vertices. The proof is by means of jellyfish graphs.

1. Introduction

The coloring problem considered in this article has restrictions on edges within distance two apart. The *distance* between two edges e and e' in a graph is the minimum k for which there is a sequence $e_0, e_1, \ldots, e_k$ of distinct edges such that $e = e_0$, $e' = e_k$, and e_{i-1} shares an end vertex with e_i for $1 \le i \le k$. A *strong edge-coloring* of a graph is a function that assigns to each edge a color such that any two edges within distance two apart receive different colors. A *color class* of a strong edge-coloring is the set of all edges using the same color. A *strong k-edge-coloring* is a strong edge-coloring using at most k colors. An *induced matching* is an edge set in which two distinct edges are of distance at least two. Finding a strong k-edge-coloring is equivalent to partitioning the edge set of the graph into k induced matchings. The *strong chromatic index* of a graph G, denoted by $\chi_s'(G)$, is the minimum k such that G admits a strong k-edge-coloring.

Strong edge-coloring was first studied by Fouquet and Jolivet [11,12] for cubic planar graphs. By a greedy algorithm, it is easy to see that $\chi_s'(G) \le 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ. Fouquet and Jolivet [11] established a Brooks type upper bound $\chi_s'(G) \le 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [26]. The following conjecture was posed by Erdős and Nešetřil [8,9] and revised by Faudree, Gyárfás, Schelp and Tuza [10]:

Conjecture 1. *If G is a graph of maximum degree Δ, then* $\chi_s'(G) \le \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.

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http://dx.doi.org/10.1016/j.disc.2015.04.031
0012-365X/

For graphs with maximum degree $\Delta = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [15] independently. For $\Delta = 4$, while Conjecture 1 says that $\chi'_s(G) \le 20$, Horák [14] obtained $\chi'_s(G) \le 23$ and Cranston [7] proved $\chi'_s(G) \le 22$. Molloy and Reed [22] proved that for large Δ every graph of maximum degree Δ has $\chi'_s(G) \le 1.998\Delta^2$ using a probabilistic method. Mahdian [19] proved that for a C_4-free graph G, $\chi'_s(G) \le (2 + o(1))\Delta^2/\ln \Delta$. Faudree, Gyárfás, Schelp and Tuza [10] proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \le \Delta^2$. They mentioned that this result could probably be improved to a linear function of the maximum degree. Brualdi and Massey [2] improved the upper bound to $\chi'_s(G) \le \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions. Nakprasit [23] proved that if G is bipartite and the maximum degree of one partite set is at most 2, then $\chi'_s(G) \le 2\Delta$. Chang and Narayanan [6] proved that $\chi'_s(G) \le 8\Delta - 6$ for chordless graphs G. This settles the above question by Faudree, Gyárfás, Schelp and Tuza [10] in the positive, since graphs with cycle lengths divisible by 4 are chordless graphs. They also established that $\chi'_s(G) \le 10\Delta - 10$ for 2-degenerate graphs G.

Strong edge-coloring on planar graphs is also extensively studied in the literature. Faudree, Gyárfás, Schelp and Tuza [10] asked whether $\chi'_s(G) \le 9$ if G is cubic planar. If this upper bound is proved to be true, it would be the best possible. Faudree, Gyárfás, Schelp and Tuza [10] used the Four-color Theorem to show that $\chi'_s(G) \le 4\Delta(G) + 4$ for any planar graph G of maximum degree Δ. They also exhibited a planar graph G whose strong chromatic index is $4\Delta(G) - 4$. Their proof also gives a consequence that $\chi'_s(G) \le 3\Delta$ for planar graphs G of girth at least 7. Chang, Montassier, Pecher and Raspaud [5] further proved that $\chi'_s(G) \le 2\Delta - 1$ for planar graphs G with large girth. Strong chromatic index for Halin graphs was first considered by Shiu, Lam and Tam [25] and then studied in [4,16,18,26]. For trees G they obtained that $\chi'_s(G) = \sigma(G)$, where

$$\sigma(G) := \max_{uv \in E(G)} \{d_G(u) + d_G(v) - 1\} \tag{1}$$

is an easy lower bound of $\chi'_s(G)$, that is,

$$\sigma(G) \le \chi'_s(G) \text{ for any graph } G. \tag{2}$$

An edge xy in a graph G is *σ-tight* if $d_G(x) + d_G(y) - 1 = \sigma(G)$. Liao [17] studied cacti, which are graphs whose blocks are cycles or complete graphs of two vertices. Notice that cacti are planar graphs and include trees. He established that for a cactus G, $\chi'_s(G) = \sigma(G)$ if the length of any cycle is a multiple of 6, $\chi'_s(G) \le \sigma(G) + 1$ if the length of any cycle is even, and $\chi'_s(G) \le \lfloor \frac{3\sigma(G)+1}{2} \rfloor$ in general. For other results on strong edge-coloring, see [3,13,20,21,24,27].

The purpose of this paper is to determine strong chromatic indices of cacti. The method is by means of jellyfish graphs to be introduced later. We first establish a decomposition theorem saying that the strong chromatic index of a graph is the maximum strong chromatic index of a block-jellyfish, which is a block together with edges with one vertex in the block and the other outside. Then we determine the strong chromatic index of a C_n-jellyfish, which is a graph obtained from the cycle C_n by attaching pendent edges to the cycle vertices.

2. Preliminary

For an integer $n \ge 3$, the *n-cycle* is the graph C_n with vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : 1 \le i \le n\}$, where $v_{n+1} = v_1$. More generally, when the indices of the vertices of an n-cycle are arithmetic expressions, they are considered to be taken modulo n.

A *cut-vertex* of a graph is a vertex whose removing results in a graph with more components than the old graph. A *block* of a graph is a maximal connected subgraph without cut-vertices in itself. Any two blocks of a graph have at most one vertex in common, and if they meet at one vertex, then it is a cut-vertex. For a block H of a graph G, any vertex $u \in V(G) - V(H)$ is adjacent to at most one vertex $v \in V(H)$, and if the vertex v exists then it is a cut-vertex of G. An *end block* is a block with exactly one cut-vertex. A *block graph* is a graph whose blocks are complete graphs. A *cactus* is a graph whose blocks are cycles or complete graphs of two vertices.

For a graph H, the *H-jellyfish* $H(p_v : v \in V(H))$ is the graph obtained from H by adding p_v new vertices adjacent to v for each vertex v in H. An edge joining a new vertex to v is called a *pendent edge* at v. A *block-jellyfish* of a graph G is the H-jellyfish H' for some block H of G, where the new vertices of H' are all vertices of $V(G) - V(H)$ having exactly one neighbor in $V(H)$. A block-jellyfish is *trivial* if it is an H-jellyfish for an end block H which is K_2, otherwise it is *non-trivial*.

Lemma 2. *If H is a subgraph of G, then $\chi'_s(H) \le \chi'_s(G)$.*

As any three consecutive edges in C_n use different colors in a strong edge-coloring, the following lemma is an easy consequence of parity checking.

Proposition 3. *If $n \ge 3$, then $\chi'_s(C_n) = 5$ for $n = 5$, $\chi'_s(C_n) = 3$ for n is a multiple of 3 and $\chi'_s(C_n) = 4$ otherwise.*

Notice that a trivial block-jellyfish H'_1 is a star; and if it is not a component, then it is a subgraph of a non-trivial block-jellyfish H'_2. By Lemma 2, $\chi'_s(H'_1) \le \chi'_s(H'_2)$.

Theorem 4. *Suppose G is a connected graph that is not a star. If G has exactly r non-trivial block-jellyfishes $G_1, G_2, \ldots, G_r$, then $\chi'_s(G) = \max_{1 \le i \le r} \chi'_s(G_i)$.*