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Frustrated triangles

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ABSTRACT

A triple of vertices in a graph is a *frustrated triangle* if it induces an odd number of edges. We study the set $F_n \subset [0, \binom{n}{3}]$ of possible number of frustrated triangles $f(G)$ in a graph G on n vertices. We prove that about two thirds of the numbers in $[0, n^{3/2}]$ cannot appear in F_n, and we characterise the graphs G with $f(G) \in [0, n^{3/2}]$. More precisely, our main result is that, for each $n \geq 3$, F_n contains two interlacing sequences $0 = a_0 \leq b_0 \leq a_1 \leq b_1 \leq \cdots \leq a_m \leq b_m \sim n^{3/2}$ such that $F_n \cap (b_t, a_{t+1}) = \emptyset$ for all t, where the gaps are $|b_t - a_{t+1}| = (n-2) - t(t+1)$ and $|a_t - b_t| = t(t-1)$. Moreover, $f(G) \in [a_t, b_t]$ if and only if G can be obtained from a complete bipartite graph by flipping exactly t edges/nonedges. On the other hand, we show, for all n sufficiently large, that if $m \in [f(n), \binom{n}{3} - f(n)]$, then $m \in F_n$ where $f(n)$ is asymptotically best possible with $f(n) \sim n^{3/2}$ for n even and $f(n) \sim \sqrt{2}n^{3/2}$ for n odd. Furthermore, we determine the graphs with the minimum number of frustrated triangles amongst those with n vertices and $e \leq n^2/4$ edges.

1. Introduction

Given a graph G on n vertices, let $t_i = t_i(G)$ denote the number of triples of vertices in G inducing i edges for $i = 0, 1, 2, 3$. One of the earliest and best-known results on colorings is due to Goodman [5], who showed that $t_0 + t_3$ is asymptotically minimised by the random graph. Goodman [6] also conjectured the maximum of $t_0 + t_3$ amongst the graphs with a given number of edges, which was later proved by Olpp [13]. More recently, Linial and Morgenstern [10] showed that every sequence of graphs with $t_0 + t_3$ asymptotically minimal is 3-universal. Hefetz and Tyomkyn [7] then proved that such sequences are 4-universal, but not necessarily 5-universal, and moreover that any sufficiently large graph H can be avoided by such a sequence.

The minimum number of triangles in a graph with a given number of edges has also been widely investigated. Erdős [3] conjectured that a graph with $\lfloor \frac{n^2}{4} \rfloor + k$ edges contains at least $k \lfloor \frac{n}{2} \rfloor$ triangles if $k < \frac{n}{2}$, which was later proved by Lovász and Simonovits [11]. More recently, Rozborov [14] determined completely the minimum number of triangles in a graph with a given number of edges using flag algebras. Some bounds for other combinations of t_0, t_1, t_2, t_3 were also given in [6,12], while similar results for three-colored graphs have recently been proved in [1,2].

In this paper, we are interested in another natural quantity $t_1 + t_3$. This study is further motivated by a phenomenon occurred in several fields of physics called *geometrical frustration* (see [4,15,16]). For example, suppose each vertex of a graph is a spin which can take only two values, say up and down, and each edge of the graph is either ferromagnetic (meaning the spins on its end points prefer to be aligned), or anti-ferromagnetic (the spins prefer to be in opposite directions). Now

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consider a cycle in the graph, and observe that there is no choice of the values of the spins satisfying the preference of every edge of the cycle if and only if there are an odd number of anti-ferromagnetic edges.

We say that a triple of vertices of a graph is a *frustrated triangle* if it induces an odd number of edges, i.e. either it contains exactly one edge, or it is a triangle. We shall write $f(G) = t_1 + t_3$ for the number of frustrated triangles in a graph G. Our aim is to study the set of possible number of frustrated triangles in a graph with n vertices,

$$F_n = \{f(G) : G \text{ is a graph on } n \text{ vertices}\}.$$

We remark that studies of similar flavor have been done, for example, in [8,9] where the object of interest is the set of possible number of colors appeared in a complete subgraph of the complete graph on $\mathbb{N}$.

Clearly, $F_n \subset [0, \binom{n}{3}]$. Note also that F_n is symmetric about the midpoint $\frac{1}{2}\binom{n}{3}$, i.e. $x \in F_n$ iff $\binom{n}{3} - x \in F_n$. This is because a triple is frustrated in G iff it is not frustrated in the complement graph $\overline{G}$, and so $f(G) + f(\overline{G}) = \binom{n}{3}$.

One might expect F_n to be the whole interval $[0, \binom{n}{3}]$. However, this is surprisingly very far from the truth. Our main result specifies subintervals of $[0, n^{3/2}]$ that are forbidden from F_n, and moreover, we characterise the graphs G with $f(G) \in [0, n^{3/2}]$.

Before we state the main theorem, let us introduce the following two sequences which play an important role throughout the paper. For $0 \le t \le n-1$, let $a_t = a_t^n$ be the number of frustrated triangles in a graph on n vertices containing t edges forming a star. For $0 \le t \le \frac{n}{2}$, let $b_t = b_t^n$ be the number of frustrated triangles in a graph on n vertices containing t edges forming a matching. It is immediate that

$$a_t = t(n-t-1) \quad \text{and} \quad b_t = t(n-2).$$

Let $t_{\max} = \max\{t : b_t < a_{t+1}\}$ be the last t such that the intervals $[a_t, b_t]$ and $[a_{t+1}, b_{t+1}]$ are disjoint. It is easy to check that

$$t_{\max} = \max\{t : t(t+1) < n-2\} = \left\lceil \sqrt{n - \frac{7}{4}} - \frac{3}{2} \right\rceil \sim \sqrt{n}$$

exists for $n \ge 3$, and

$$0 = a_0 = b_0 < a_1 = b_1 < a_2 < b_2 < \cdots < a_{t_{max}} < b_{t_{max}} < a_{t_{max}+1} \sim n^{3/2}.$$

We can now state the main result.

Theorem 1. *Let G be a graph on $n \ge 3$ vertices.*

(i) *If $f(G) < a_{t_{max}+1}$ then $f(G) \in [a_t, b_t]$ for a unique $t \le t_{max}$.*
(ii) *If $t \le t_{max}$ then $f(G) \in [a_t, b_t]$ iff G can be obtained from a complete bipartite graph on n vertices by flipping exactly t pairs of vertices.*

Here, to *flip* a pair of vertices uv means to change its edge/nonedge status, i.e. if uv is an edge, make it a nonedge, and vice versa. We remark that Theorem 1 part (i) is equivalent to the statement: $f(G) \notin (b_t, a_{t+1})$ for all $t \ge 0$. Note also that the intervals have lengths

$$\left|(b_t, a_{t+1})\right| = (n-2) - t(t+1) \quad \text{and} \quad \left|[a_t, b_t]\right| = t(t-1).$$

Theorem 1 is complemented by the following result. Observe that $a_{t_{max}+1} = (t_{max}+1)(n-t_{max}-2) \sim n^{3/2}$ and so Theorem 1 deals with the case $f(G) \lesssim n^{3/2}$. Since F_n is symmetric, we automatically have a corresponding result for $f(G) \gtrsim \binom{n}{3} - n^{3/2}$. On the other hand, we prove that every number, up to parity condition, in the large central part of $[0, \binom{n}{3}]$ is realisable as $f(G)$ for some G. Note that if n is even, then $f(G)$ must also be even, since adding an edge to a graph changes the 'frustration status' of exactly $n-2$ triples.

Theorem 2. (i) *If n is even and sufficiently large then F_n contains every even integer between $n^{3/2} + O(n^{5/4})$ and $\binom{n}{3} - (n^{3/2} + O(n^{5/4}))$, and this is best possible up to the second order term.*
(ii) *If n is odd and sufficiently large then F_n contains every integer between $\sqrt{2}n^{3/2} + O(n^{5/4})$ and $\binom{n}{3} - (\sqrt{2}n^{3/2} + O(n^{5/4}))$, and this is best possible up to the second order term.*

Let us change the direction and turn to the following related natural question. Given the number of vertices and the number of edges, which graphs maximise/minimise the number of frustrated triangles? The method we develop to prove Theorem 1 allows us to partially answer this question. Before we state the result, we shall define some necessary notations. For $0 \le x \le \frac{n}{2}$, let $c_x^n = x(n-x)$ be the number of edges in the complete bipartite graph $K_{x,n-x}$. For an integer e, let $g(e) = \min\{|e - c_x| : 0 \le x \le \frac{n}{2}\}$ be the distance from e to the sequence (c_x). We are able to determine the minimal graphs when the number of edges is at most $\frac{n^2}{4}$.

Theorem 3. *If G is a graph with n vertices and e edges then $f(G) \ge a_{g(e)}$. Moreover, this bound can be achieved when $e \le \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n-1}{2} \rfloor - 1$: in this case the extremal graphs are obtained from a complete bipartite graph by deleting or adding $g(e)$ edges forming a star.*