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Note

Equality of distance packing numbers

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ABSTRACT

We characterize the graphs for which the independence number equals the packing number. As a consequence we obtain simple structural descriptions of the graphs for which (i) the distance-k-packing number equals the distance-$2k$-packing number, and (ii) the distance-k-matching number equals the distance-$2k$-matching number. This last result considerably simplifies and extends previous results of Cameron and Walker (2005). For positive integers k_1 and k_2 with $k_1 < k_2$ and $\lceil(3k_2+1)/2\rceil \leq 2k_1+1$, we prove that it is NP-hard to determine for a given graph whether its distance-k_1-packing number equals its distance-k_2-packing number.

1. Introduction

Induced matchings in graphs were introduced by Stockmeyer and Vazirani [12] as a variant of ordinary matchings. While the structure and algorithmic properties of ordinary matchings are well understood [11], induced matchings are algorithmically very hard [2,12,7]. Many efficient algorithms for finding maximum induced matchings exploit the fact that induced matchings correspond to independent sets of the square of the line graph [1,4,6,3,9]. In [10] Kobler and Rotics showed that the graphs where the matching number and the induced matching number coincide can be recognized efficiently. Their result was extended by Cameron and Walker [5] who gave a complete structural description of these graph. In [8] we generalized some results from [10,5] to distance-k-matchings and simplified the original proofs. In the present paper we present much more general results systematically exploiting the above-mentioned relation between matchings and independent sets in line graphs. Our main result is a very simple characterization of the graphs for which the independence number equals the packing number. An immediate consequence of this result is a complete structural description of the graphs for which the distance-k-matching number equals the distance-$2k$-matching number. It follows immediately that such graphs can be recognized by a very simple efficient algorithm. We establish further results relating distance packing numbers and discuss related open problems.

Before we proceed to the results, we recall some terminology. We consider finite, simple, and undirected graphs. Let G be a graph. A set P of vertices of G is a *k-packing of G* for some positive integer k if every two distinct vertices in P have distance more than k in G. The *k-packing number $\rho_k(G)$ of G* is the maximum cardinality of a k-packing of G, and a k-packing of cardinality $\rho_k(G)$ is *maximum*. Using this terminology, independent sets correspond to 1-packings and the independence number $\alpha(G)$ coincides with $\rho_1(G)$. We denote the *line graph of G* by $L(G)$ and the *kth power* of G for some positive integer k by G^k. Since matchings of G correspond to independent sets of $L(G)$, the matching number $\nu(G)$ equals $\rho_1(L(G))$. Similarly, since induced matchings of G correspond to 2-packings of $L(G)$, the induced matching number $\nu_2(G)$ equals $\rho_2(L(G))$. More generally, a set M of edges of G is a *k-matching of G* if it is a k-packing of $L(G)$. The *k-matching number $\nu_k(G)$* and *maximum k-matchings* are defined in the obvious way. Clearly, a set P is a k_1-packing of G^{k_2} for some positive integers k_1 and k_2 if and

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only it is a k_1k_2-packing of G, that is, $\rho_{k_1}(G^{k_2}) = \rho_1(G^{k_1k_2})$. A vertex u of G is *simplicial* if $N_G[u]$ is complete. Two distinct vertices u and v of G are *twins* if $N_G[u] = N_G[v]$. Let $S(G)$ be the set of simplicial vertices of G. Let $\mathcal{S}(G)$ be the partition of $S(G)$ where two simplicial vertices belong to the same partite set if and only if they are twins. A *transversal of* $\mathcal{S}(G)$ is a set of simplicial vertices that contains exactly one vertex from each partite set of the partition $\mathcal{S}(G)$. Note that the subgraph of G induced by $S(G)$ is a union of cliques, and that $\mathcal{S}(G)$ is the collection of the vertex sets of these cliques. In particular, every transversal of $\mathcal{S}(G)$ is independent.

2. Results

We immediately proceed to the characterization of the graphs for which the independence number equals the packing number.

Theorem 1. *A graph G satisfies $\rho_1(G) = \rho_2(G)$ if and only if*

(i) *a set of vertices of G is a maximum 2-packing if and only if it is a transversal of $\mathcal{S}(G)$, and*
(ii) *for every transversal P of $\mathcal{S}(G)$, the sets $N_G[u]$ for u in P partition $V(G)$.*

Proof. Let G be a graph.

In order to prove the sufficiency, let G satisfy (i) and (ii). Let P be a transversal of $\mathcal{S}(G)$. By (i), we have $|P| = \rho_2(G)$. By (ii) and since $P \subseteq S(G)$, we obtain that $\{N_G[u] : u \in P\}$ is a partition of $V(G)$ into complete sets. Since every 1-packing contains at most one vertex from each complete set, this implies $\rho_1(G) \leq |P|$. Since $\rho_1(G) \geq \rho_2(G)$, it follows $\rho_1(G) = \rho_2(G)$.

In order to prove the necessity, let G satisfy $\rho_1(G) = \rho_2(G)$. Let P be a maximum 2-packing. If some vertex u in P has two non-adjacent neighbors v and w, then $(P\setminus\{u\}) \cup \{v, w\}$ is a 1-packing with more vertices than P, which is a contradiction. Hence all vertices in P are simplicial. Since no two vertices in P are adjacent, the set P is contained in some transversal Q of $\mathcal{S}(G)$. Since Q is a 1-packing, we obtain $\rho_2(G) = |P| \leq |Q| \leq \rho_1(G) = \rho_2(G)$, that is, $P = Q$, which implies in particular that P is a transversal of $\mathcal{S}(G)$. If $V(G)\setminus \bigcup_{u\in P} N_G[u]$ contains a vertex v, then $P \cup \{v\}$ is 1-packing with more vertices than P, which is a contradiction. Hence $\{N_G[u] : u \in P\}$ is a partition of $V(G)$ into complete sets. Since for every transversal P' of $\mathcal{S}(G)$, the partition $\{N_G[u'] : u' \in P'\}$ equals the partition $\{N_G[u] : u \in P\}$, it follows that every transversal of $\mathcal{S}(G)$ is a maximum 2-packing. Altogether, (i) and (ii) follow. □

By considering suitable powers of the underlying graph, we obtain the following.

Corollary 2. *A graph G satisfies $\rho_k(G) = \rho_{2k}(G)$ for some positive integer k if and only if*

(i) *a set of vertices of G is a maximum $2k$-packing if and only if it is a transversal of $\mathcal{S}(G^k)$, and*
(ii) *for every transversal P of $\mathcal{S}(G^k)$, the sets $N_{G^k}[u]$ for u in P partition $V(G)$.*

By Corollary 2, it is algorithmically very easy to recognize the graphs G with $\rho_k(G) = \rho_{2k}(G)$.

In view of Theorem 1 and Corollary 2, it makes sense to consider the equality of distance packing numbers $\rho_{k_1}(G)$ and $\rho_{k_2}(G)$ where $k_1 < k_2$ are positive integers that do not satisfy $k_2 = 2k_1$. Our next observation shows that for $k_2 > 2k_1$ such graphs are not very interesting.

Observation 3. *If k_1 and k_2 are positive integers with $k_2 > 2k_1$ and G is a connected graph with $\rho_{k_1}(G) = \rho_{k_2}(G)$, then $\rho_{k_1}(G) = \rho_{k_2}(G) = 1$.*

Proof. Let G be a graph that satisfies $\rho_{k_1}(G) = \rho_{k_2}(G)$. Let P be a maximum k_2-packing. For a contradiction, we assume that P has more than one element. Let u be a vertex in P. Since P has more than one element, there is a vertex v at distance $k_1 + 1$ from u. Since $k_2 + 1 \geq 2(k_1 + 1)$, every vertex in P has distance more than k_1 from v. Now $P \cup \{v\}$ is a k_1-packing, which is a contradiction. This completes the proof. □

Now we consider the case $k_1 < k_2 < 2k_1$ and show that already the smallest possible choice, $k_1 = 2$ and $k_2 = 3$, leads to graphs that will most likely not have a nice structural description.

Theorem 4. *It is NP-hard to determine for a given graph G whether $\rho_2(G) = \rho_3(G)$.*

Proof. We describe a reduction from 3SAT to the considered problem. Therefore, let f be a 3SAT instance with m clauses $C_1, \ldots, C_m$ over n boolean variables $x_1, \ldots, x_n$. We construct a graph G whose order is polynomially bounded in terms of n and m such that f is satisfiable if and only if $\rho_2(G) = \rho_3(G)$. For every variable x_i, we create a cycle $G(x_i) : x_i\bar{x}_i x_i' \bar{x}_i' x_i$ of length 4 as shown in the left of Fig. 1. For every clause C_j, we create a copy $G(C_j)$ of the graph in the right of Fig. 1 and denote its vertices as explained in the caption. All graphs $G(x_i)$ and $G(C_j)$ created so far are disjoint. For every clause C with literals x, y, and z, we create the three edges $x'(C)x'$, $y'(C)y'$, and $z'(C)z'$. If, for example, $C_1 = x_1 \vee \bar{x}_2 \vee \bar{x}_4$, then these are the edges $x_1'(C)x_1'$, $\bar{x}_2'(C)\bar{x}_2'$, and $\bar{x}_4'(C)\bar{x}_4'$ as shown in Fig. 2. This completes the description of G. It is easy to verify that $\rho_2(G(x_i)) = 1$ and $\rho_2(G(C_j)) = 2$, which implies that $\rho_2(G) \leq n+2m$. Since $\{a(C_j) : j \in [m]\} \cup \{b(C_j) : j \in [m]\} \cup \{x_i : i \in [n]\}$ is a 2-packing of cardinality $n + 2m$, we obtain $\rho_2(G) = n + 2m$. It remains to prove that f is satisfiable if and only if $\rho_3(G) = n + 2m$.