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Centralization of transmission in networks

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ABSTRACT

Freeman's centralization (Freeman, 1978) for a given centrality index is a measure of how central a vertex is regarding to how central all the other vertices are with respect to the given index. The transmission of a vertex v in a graph G is equal to the sum of distances between v and all other vertices of G. In this paper we study the centralization of transmission, in particular, we determine the graphs on n vertices which attain the maximum or minimum value. Roughly, the maximizing graphs are comprised of a path which has one end glued to a clique of similar order. The minimizing family of extremal graphs consists of three paths of almost the same length, glued together in one end-vertex. We conclude the paper with some problems for possible further work.

1. Introduction

The *transmission* of a vertex $v \in V(G)$ (in some literature also called *farness* or *vertex Wiener value*) is defined as the sum of the lengths of all shortest paths between a chosen vertex and all other vertices in G, i.e.

$$W(v) = \sum_{u \in V(G)\setminus\{v\}} d_G(u, v).$$

Using transmission, one can define a well-known topological index that we also use later in the definition of transmission centralization. Let us now briefly describe this index. The Wiener index $W(G)$, introduced by Wiener [14], is a graph index defined for connected graph G as the sum of the lengths of shortest paths between all unordered pairs of vertices in G, formally

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} W(v).$$

It is the oldest topological index related to molecular branching and based on its success, many other topological indices correlated to distance matrix of chemical graphs have been developed subsequently to Wiener's work. Wiener index was at first used for predicting the boiling points of paraffins [14], but later a strong correlation between Wiener index and other chemical or physical properties of a compound was found, such as critical points in general [13], the density, surface tension, and viscosity of compounds liquid phase [11] and the van der Waals surface area of the molecule [6]. There are some recent papers on Wiener index of trees [7], common neighborhood graphs [3,8] and line graphs [15]. Finding graph extremals for

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Wiener index and its derivatives is nicely summarized in a recent survey by Gutman et al. [16]. It is easy to conclude that among connected graphs on n vertices, minimal and maximal values of Wiener index are $\binom{n}{2}$ and $\binom{n+1}{3}$ obtained at K_n and P_n, respectively. In the class of trees, both extremal graphs are S_n and P_n with Wiener values $(n-1)^2$ and $\binom{n+1}{3}$, respectively. These and many other bounds for the Wiener index are presented in [16,9].

In graph theory, centrality refers to indices which identify the most important/central vertices within a graph. Those most commonly used measures are betweenness centrality, closeness centrality, degree and eccentricity. Several aspects of correlation between Wiener index and betweenness centrality are presented in the paper of Caporossi et al. [2], where authors assign betweenness-related weights to edges of a graph that sum up to its Wiener index. For graphs with fixed order they also find extremal graphs for lower and upper bounds of betweenness centrality. A theorem of Wiener [14], shows how the Wiener index of a tree is decomposed into (easily calculable) edge-contributions. In [12], authors introduce a vertex-version of this theorem for general graphs by using the correlation of Wiener index to betweenness centrality.

The *centralization* of a graph is a measure of how central its most central vertex is with respect to how central all the other vertices are. The general definition of centralization for graphs, proposed by Freeman [5], assigns a centralization measure F_1 to any existing centrality measure F, i.e.

$$F_1(v) = \sum_{u \in V(G)} (F(v) - F(u)) .$$

Centralization measures calculate the sum of differences in centrality between the most central vertex in a graph and all other vertices, thus every centrality measure can have its own centralization measure. In 2006, Butts [1] studied the extremal values of degree centralization among all graphs on n vertices. Everett, Dankelmann and Sinclair [4] also studied extremal values for centralization in two-mode graphs. *Transmission centralization* of a vertex $v \in V(G)$ is obtained by applying Freeman's notion of the centralization to Wiener index, formally

$$W_1(v) = \sum_{u \in V(G)\setminus\{v\}} (W(v) - W(u)) = n \cdot W(v) - 2W(G), \tag{1}$$

where $W(G)$ is the Wiener index of a graph G. In order to compare centralization values of graphs with different sizes, Freeman in the definition of centralization originally used a normalized formula, dividing expression (1) by the theoretically largest such sum of differences in any graph from the given class of graphs [5]. Since in this paper, the size of our graph is of constant size, we omit the normalizing denominators.

Among all graphs on n vertices $\mathcal{G}_n$, those that achieve maximum or minimum Wiener centralization value will be called *extremal graphs.* In the paper, we assume that $n > 1$. Instead of $W(v)$ and $W_1(v)$ we will sometimes also write $W(v, G)$ and $W_1(v, G)$, to emphasize the underlying graph we are dealing with. The *eccentricity* of a vertex w is defined as $\max_{v \in V(G)} d_G(w, v)$.

The paper is structured as follows. In section two, we present the structure of graphs that attain maximal Wiener centralization while in section three we focus on the lower bound. In section four we conclude the paper with some ideas for possible future work.

2. Upper bound of transmission centralization

In lemmas that follow, we assume that G is a connected graph on n vertices that maximizes transmission centralization among all graphs in $\mathcal{G}_n$. Also, let $w \in V(G)$ be a vertex at which transmission centralization is maximized and let d be the eccentricity of the vertex w. By the choice of w, it is easy to see that for any $t \in V(G)$ we have

$$W(w, G) \geq W(t, G) \quad \text{and} \quad W_1(w, G) \geq 0. \tag{2}$$

Let L_i be the set of vertices at distance i from w in G, i.e. $L_i = \{v \in G;\ d_G(v, w) = i\}$ and let $l_i = |L_i|$. We say that L_i is the ith layer from w. Note that $L_0 = \{w\}$.

Lemma 1. *Let i be a non-negative integer. Then vertices in L_i and L_{i+1} induce a complete graph.*

Proof. Assume that there exist two non-connected vertices $u, v \in L_i \cup L_{i+1}$ that violate the claim of this lemma. It is easy to see that adding an edge uv does not affect the value of $W(w)$. On the other hand, introducing the edge uv (or any new edge) always decreases Wiener index of the whole graph. Therefore, introducing the edge uv increases expression (1), a contradiction. □

An example of the connected members of $\mathcal{G}_4$ that are consistent with Lemma 1 can be observed on Fig. 1. A layer is *trivial* if it is comprised of one vertex.

Lemma 2. *Layers $L_1, L_2, \ldots, L_{\lfloor n/2 \rfloor - 1}$ are trivial.*

Proof. Let $s = \left\lfloor \frac{n}{2} \right\rfloor$. We proceed with contradiction, assuming that some of these layers in G is non-trivial. We prove the claim by introducing an operation that iteratively transform nearest $s-1$ vertices from w into a path, increasing its transmission centralization at each step.