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Multivariate Stirling polynomials of the first and second kind

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ABSTRACT

Two doubly indexed families of homogeneous and isobaric polynomials in several indeterminates are considered: the (partial) exponential Bell polynomials $B_{n,k}$ and a new family $S_{n,k} \in \mathbb{Z}[X_1, \ldots, X_{n-k+1}]$ such that $X_1^{-(2n-1)} S_{n,k}$ and $B_{n,k}$ obey an inversion law which generalizes that of the Stirling numbers of the first and second kind. Both polynomial families appear as Lie coefficients in expansions of certain derivatives of higher order. Substituting $D^j(\varphi)$ (the jth derivative of a fixed function φ) in place of the indeterminates X_j shows that both $S_{n,k}$ and $B_{n,k}$ are differential polynomials depending on φ and on its inverse $\overline{\varphi}$, respectively. Some new light is shed thereby on Comtet's solution of the Lagrange inversion problem in terms of the Bell polynomials. According to Haiman and Schmitt that solution is essentially the antipode on the Faà di Bruno Hopf algebra. It can be represented by $X_1^{-(2n-1)} S_{n,1}$. Moreover, a general expansion formula that holds for the whole family $S_{n,k}$ ($1 \leq k \leq n$) is established together with a closed expression for the coefficients of $S_{n,k}$. Several important properties of the Stirling numbers are demonstrated to be special cases of relations between the corresponding polynomials. As a non-trivial example, a Schlömilch-type formula is derived expressing $S_{n,k}$ in terms of the Bell polynomials $B_{n,k}$, and vice versa.

1. Introduction

1.1. Background and problem

It is well-known that a close connection exists between iterated differentiation and Stirling numbers [11,19]. Let $s_1(n, k)$ denote the signed Stirling numbers of the first kind, $s_2(n, k)$ the Stirling numbers of the second kind, and D the operator d/dx. Then, for all positive integers n, the nth iterate $(xD)^n$ can be expanded into the sum

$$(xD)^n = \sum_{k=1}^{n} s_2(n, k) x^k D^k. \tag{1.1}$$

An expansion in the reverse direction is also known to be valid (see, e. g., [11, p. 197], or [19, p. 45]):

$$D^n = x^{-n} \sum_{k=1}^{n} s_1(n, k)(xD)^k. \tag{1.2}$$

Let us first look at (1.1). The appearance of the Stirling numbers may be combinatorially explained as follows. Observing

$$(xD)^n f(x) = (D^n(f \circ \exp))(\log x),$$

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we can use the classical higher order chain rule (named after Faà di Bruno; cf. [10,11], [12, pp. 52,481]) to calculate the nth derivative of the composite function $f \circ g$:

$$(f \circ g)^{(n)}(x) = \sum_{k=1}^{n} B_{n,k}(g'(x), \dots, g^{(n-k+1)}(x)) \cdot f^{(k)}(g(x)), \tag{1.3}$$

where $B_{n,k} \in \mathbb{Z}[X_1, \dots, X_{n-k+1}]$, $1 \le k \le n$, is the (partial) exponential Bell polynomial [2]

$$B_{n,k}(X_1, \dots, X_{n-k+1}) = \sum_{r_1, r_2, \dots} \frac{n!}{r_1! r_2! \cdots (1!)^{r_1} (2!)^{r_2} \cdots} X_1^{r_1} X_2^{r_2} \cdots, \tag{1.4}$$

the sum to be taken over all non-negative integers $r_1, r_2, \dots, r_{n-k+1}$ such that $r_1 + r_2 + \cdots + r_{n-k+1} = k$ and $r_1 + 2r_2 + \cdots + (n-k+1)r_{n-k+1} = n$. The coefficient in $B_{n,k}$ counts the number of partitions of n distinct objects into k blocks (subsets) with r_j blocks containing exactly j objects ($1 \le j \le n-k+1$). Therefore, the sum of these coefficients is equal to the number $s_2(n, k)$ of all such partitions. So we have $B_{n,k}(x, \dots, x) = s_2(n, k)x^k$. Evaluating $(f \circ \exp)^{(n)}(\log x)$ by (1.3) then immediately gives the right-hand side of (1.1).

Question. Is there an analogous way of also interpreting (1.2) by substituting jth derivatives in place of the indeterminates X_j of some polynomial $S_{n,k} \in \mathbb{Z}[X_1, \dots, X_{n-k+1}]$ whose coefficients add up to $s_1(n, k)$?

The main purpose of the present paper is to give a positive and comprehensive answer to this question including recurrences, a detailed study of the inversion relationship between the families $B_{n,k}$ and $S_{n,k}$, as well as fully explicit formulas (with some applications to Stirling numbers and Lagrange inversion).

The issue turns out to be closely related to the problem of generalizing (1.1), that is, finding an expansion for the operator $(\theta D)^n$ ($n \ge 1$, θ a function of x). Note that, in the case of scalar functions, $(\theta D)f$ is the *Lie derivative* of f with respect to θ. Several authors have dealt with this problem. In [4] and [17] a polynomial family $F_{n,k} \in \mathbb{Z}[X_0, X_1, \dots, X_{n-k}]$ has been defined[1] by differential recurrences, and shown to comply with $(\theta D)^n = \sum_{k=1}^{n} F_{n,k}(\theta, \theta', \dots, \theta^{(n-k)})D^k$. Comtet [4] has tabulated $F_{n,k}$ up to $n = 7$ and proved that $F_{n,k}(x, \dots, x) = c(n, k)x^n$, where $c(n, k) := |s_1(n, k)|$ denotes the signless Stirling numbers of the first kind ('cycle numbers' following the terminology in [13]). Since however all coefficients of $F_{n,k}$ are positive, $F_{n,k}$ does not appear to be a suitable companion for $B_{n,k}$ with regard to the desired inversion law.

Todorov [27,28] has studied the above Lie derivation with respect to a function θ of the special form $\theta(x) = 1/\varphi'(x)$, $\varphi'(x) \neq 0$. His main results in [27] ensure the existence of $S_{n,k} \in \mathbb{Z}[X_1, \dots, X_{n-k+1}]$ such that, taking $A_{n,k} := X_1^{-(2n-1)} S_{n,k}$, we have the following expansion:

$$\left(\frac{D}{\varphi'(x)}\right)^n f(x) = \sum_{k=1}^{n} A_{n,k}(\varphi'(x), \dots, \varphi^{(n-k+1)}(x)) \cdot f^{(k)}(x). \tag{1.5}$$

While differential recurrences for $A_{n,k}$ can readily be derived from (1.5) (cf. [27, eq. (27)], or a slightly modified version in [28, Theorem 2]), a simple representation for $S_{n,k}$ – as is (1.4) for $B_{n,k}$ – was still lacking (up to now). Todorov [27, p. 224] erroneously believed that the (somewhat cumbersome) 'explicit' expression in [4] for the coefficients of $F_{n,k}$ would give the coefficients of $S_{n,k}$ as well. Also the determinantal form presented for $S_{n,k}$ in [27, Theorem 6] may only in a modest sense be regarded as an 'explicit formula'.

Nevertheless, Todorov's choice ($\theta = 1/\varphi'$) eventually proves to be a crucial idea, since it reveals that $A_{n,k}$ (and so $S_{n,k}$) is connected with the classical Lagrange problem of computing the coefficients $\bar{f}_n$ of the compositional inverse $\bar{f}$ of a given series $f(x) = \sum_{n \ge 1}(f_n/n!)x^n$, $f_1 \neq 0$. Theorem 9 in [27] provides a method to achieve this by means of the determinantal expression for $S_{n,k}$ mentioned above. As we shall see later, $\bar{f}_n$ can be given by far a simpler and fairly concise form:

$$\bar{f}_n = A_{n,1}(f_1, \dots, f_n). \tag{1.6}$$

On the other hand, Comtet [5] has found an inversion formula that expresses $\bar{f}_n$ through the (partial) exponential Bell polynomials:

$$\bar{f}_n = \sum_{k=1}^{n-1} (-1)^k f_1^{-n-k} B_{n+k-1,k}(0, f_2, \dots, f_n). \tag{1.7}$$

This result has been shown by Haiman and Schmitt [8] and Schmitt [20] to provide essentially both a combinatorial representation and a cancellation-free computation of the antipode on a Faà di Bruno Hopf algebra (a subject much attention has been payed to in quantum mechanics because of its applications to renormalization theory; cf. [6]). Combining (1.6) with

[1] Here I write $F_{n,k}$ instead of $A_{n,k}$ (cf. [4]) in order to obviate misunderstandings, as $A_{n,k}$ will be used throughout the present paper to denote the 'Lie coefficients' according to Todorov (corresponding to his $B_{n,k}$ in [27] as well as to his $L_{n,k}$ in [28]).