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Note

Edge coloring multigraphs without small dense subsets

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A B S T R A C T

One consequence of a long-standing conjecture of Goldberg and Seymour about the chromatic index of multigraphs would be the following statement. Suppose G is a multigraph with maximum degree Δ, such that no vertex subset S of odd size at most Δ induces more than $(\Delta+1)(|S|-1)/2$ edges. Then G has an edge coloring with $\Delta+1$ colors. Here we prove a weakened version of this statement.

1. Introduction

In this note we study edge colorings of (loopless) multigraphs. We use the standard notation $\chi'(G)$ to denote the chromatic index of the multigraph G, that is, the smallest number of matchings needed to partition the edge set of G. It is clear that the maximum degree $\Delta(G)$ is a lower bound for $\chi'(G)$ for every graph G. The classical upper bounds for $\chi'(G)$ are $\chi'(G) \leq 3\Delta(G)/2$ (Shannon's Theorem [15]) and $\chi'(G) \leq \Delta(G) + \mu(G)$ (Vizing's Theorem [18]), where $\mu(G)$ denotes the maximum edge multiplicity of G.

For a multigraph G, a subset $S \subseteq V(G)$, and a subgraph $H \subseteq G$, we denote by $G[S]$ the subgraph induced by S, by $\|H\|$ the number of edges in H, and by $|H|$ the number of vertices in H. We also set $G[H] = G[V(H)]$ and $\|S\| = \|G[S]\|$. Let $\rho(S)$ be the quantity $\frac{\|S\|}{\lfloor |S|/2 \rfloor}$. The parameter $\rho(G)$ is defined by

$$\rho(G) = \max\{\rho(S) : S \subseteq V(G)\}.$$

Then $\lceil \rho(G) \rceil$ is a lower bound on $\chi'(G)$, since for a set S on which $\rho(G)$ is attained, each matching in $G[S]$ has size at most $\lfloor |S|/2 \rfloor$ and therefore at least $\lceil \frac{\|S\|}{\lfloor |S|/2 \rfloor} \rceil$ colors are needed to color the edges of $G[S]$. On the other hand, when $\rho(G) \geq \Delta(G)$ the chromatic index can also be bounded above in terms of $\lceil \rho(G) \rceil$. Kahn [7] gave the bound $\chi'(G) \leq (1 + o(1))\lceil \rho(G) \rceil$, which was recently improved by Plantholt [10] to

$$\chi'(G) \leq \left(1 + \frac{\log_{3/2} \lceil \rho(G) \rceil}{\lceil \rho(G) \rceil}\right) \lceil \rho(G) \rceil.$$

The focus of this paper is the long-standing conjecture due to Goldberg [3] (see also [4]) and independently Seymour [14] which states that the chromatic index of G should be essentially determined by either $\rho(G)$ or $\Delta(G)$.

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Conjecture 1. *For every multigraph G*

$$\chi'(G) \le \max\{\Delta(G)+1, \lceil \rho(G) \rceil\}.$$

Goldberg [4] also proposed the following sharp version for multigraphs with $\rho(G) \le \Delta(G) - 1$.

Conjecture 2. *For every multigraph G, if* $\rho(G) \le \Delta(G) - 1$ *then* $\chi'(G) = \Delta(G)$.

Conjecture 1 implies that if $\chi'(G) > \Delta + k$, $k \ge 1$, then G must contain a set S of vertices for which $\rho(S) > \Delta + k$, certifying this inequality. Thus S induces a very dense subgraph in G. As $\|S\| \le \Delta(G)|S|/2$, if $|S|$ is even then $\rho(S) \le \Delta(G)$; so $|S|$ is odd and $\rho(S) \le \Delta(G)|S|/(|S|-1) = \Delta(G) + \Delta(G)/(|S|-1)$. We say S is *small* in the sense that its size depends only on Δ and not on the number of vertices of G. In particular $|S| \le \Delta(G)$. Conjecture 2 gives a similar statement for $k = 0$, but the corresponding set S need not be small.

We can therefore think of Conjecture 1 as providing structural information about multigraphs for which $\chi'(G) > \Delta + 1$, namely, that they must contain small sets S that are very dense. Our aim in this note is to prove a result of this form. Unfortunately we cannot make such a conclusion about all G with $\chi'(G) > \Delta + 1$, but we show that when k is bounded below by a logarithmic function of Δ then a structural result of this type for multigraphs G satisfying $\chi'(G) > \Delta + k$ is possible.

Conjecture 1 has inspired a significant body of work, with contributions from many researchers, see for example [16] or [6] for an overview. Here we mention just the results that directly relate to this note. The best known approximate version is as follows, due to Scheide [11] (independently proved by Chen, Yu and Zang [1], see also [12] and [2]), who proved that the conjecture is true when $\lceil \rho(G) \rceil \ge \Delta + \sqrt{\frac{\Delta-1}{2}}$.

Theorem 3. *For every multigraph G*

$$\chi'(G) \le \max\left\{\Delta(G) + \sqrt{\frac{\Delta(G)-1}{2}}, \lceil \rho(G) \rceil\right\}.$$

Since $\lceil \rho(S) \rceil > \Delta + \sqrt{\frac{\Delta-1}{2}}$ implies $|S| < \sqrt{\frac{2\Delta^2}{\Delta-1}} + 1$, the following corollary about multigraphs without small dense subsets is implied by Theorem 3.

Corollary 4. *Let G be a multigraph with maximum degree Δ. If* $\lceil \rho(S) \rceil \le \Delta + \sqrt{\frac{\Delta-1}{2}}$ *for every* $S \subseteq V(G)$ *with* $|S| < \sqrt{\frac{2\Delta^2}{\Delta-1}} + 1$ *then* $\chi'(G) \le \Delta + \sqrt{\frac{\Delta-1}{2}}$.

The main theorem of this note states that if the density of small vertex subsets S is restricted somewhat further then a substantially better upper bound can be given for $\chi'(G)$, in which the quantity $\sqrt{\frac{\Delta-1}{2}}$ in the conclusion of Corollary 4 is replaced by a logarithmic function of Δ. It can also be viewed as a weakened version of the statement of Conjecture 2.

Theorem 5. *Let G be a multigraph with maximum degree Δ, and let ε be given where* $0 < \varepsilon < 1$. *Let* $k = \lfloor \log_{1+\varepsilon} \Delta \rfloor$. *If* $\rho(S) \le (1-\varepsilon)(\Delta + k)$ *for every* $S \subseteq V(G)$ *with* $|S| < \Delta/k + 1$ *then* $\chi'(G) \le \Delta + k$.

For example, this implies that $\chi'(G) < \Delta + 101 \log \Delta$ unless G contains a set S of vertices with $|S| < \frac{\Delta}{100 \log \Delta}$ with density parameter $\rho(S) > 0.99(\Delta + 100 \log \Delta)$.

Our proof uses the technique of Tashkinov trees, developed by Tashkinov in [17]. In the next section we give a brief introduction to this technique together with the main tools we use, including our main technical lemma, Lemma 8. The proof of Theorem 5 appears in Section 3.

2. Tools

The method of Tashkinov trees, due to Tashkinov [17], is a sophisticated generalization of the method of alternating paths. It is based on an earlier approach from [8]. See [16] for a comprehensive account of this technique.

Let G be a multigraph with $\chi'(G) \ge \Delta + 2$, and let ϕ be a partial edge coloring of G that uses at most $\chi' - 1$ colors. We say ϕ is a *t-coloring* if the codomain of ϕ is $\{1, \dots, t\}$. We normally assume ϕ is *maximal*, that is, the maximum possible number of edges of G are colored by ϕ. For a vertex v of G, color α is said to be *missing* at v if no edge incident to v is colored α by ϕ. Let $T = (p_0, e_0, p_1, \dots, e_n - 1, p_n)$ be a sequence of distinct vertices p_i and edges e_i of G, such that the vertices of each e_i are p_{i+1} and p_r for some $r \in \{0, \dots, i\}$. Observe that the vertices and edges of T form a tree. We say that T is a *Tashkinov tree* with respect to ϕ if e_0 is uncolored, and for all $i > 0$, the color $\phi(e_i)$ is missing at p_j for some $j < i$. Thus T is a Tashkinov tree if its first edge is uncolored, and each subsequent edge is colored with a color that is missing at some previous vertex. The key property of Tashkinov trees is captured in the following theorem, due to Tashkinov [17].