Discrete Mathematics 338 (2015) 2514–2522

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Towards size reconstruction from fewer cards

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ARTICLE INFO

Article history:
Received 3 September 2014
Received in revised form 28 May 2015
Accepted 21 June 2015
Available online 11 July 2015

Keywords:
Size reconstruction
Card
Vertex-deleted subgraph

ABSTRACT

It is proved that if two graphs of order n have $n-p$ cards (vertex-deleted subgraphs) in common, where $p \geqslant 3$, and n is large enough compared with p, then the numbers of edges in the two graphs differ by at most $p-2$. This is a modest but nontrivial improvement of the easy result that these numbers differ by at most p.

1. Introduction

Let G be a graph with vertex-set $V(G)$, edge-set $E(G)$, and minimum degree $\delta(G)$. A *card* or *vertex-deleted subgraph* of G is a graph of the form $G-v$, where $v \in V(G)$. Two graphs G and H of order n have $n-p$ *cards in common* if their vertices can be labeled as $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$, respectively, so that $G-v_i \cong H-u_i$ $(i = 1, \ldots, n-p)$. The well-known Reconstruction Conjecture is that if two graphs of order $n \geqslant 3$ have n cards in common, then they are isomorphic (see, for example, [1,2]).

It is of interest to explore what one can deduce about two graphs if they have $n-p$ cards in common, especially when n is large in terms of p. Myrvold [3] proved the following result.

Theorem 1.1 *([3]). If G and H are graphs of order $n \geqslant 7$ with $n-1$ cards in common, then G and H have the same degree sequence.*

The condition $n \geqslant 7$ here is sharp: for $n = 6$, there is exactly one pair of graphs forming a counterexample, found by Myrvold and shown in Fig. 1(a). (Note that each of the graphs in Fig. 1(a) is isomorphic to the complement of the other; if this were not so, then their complements would form another counterexample.)

The proof of Theorem 1.1 is mainly devoted to proving that G and H have the same number of edges. It is easy to see that if G and H have $n-1$ cards in common and the same number of edges then they have the same degree sequence: see Lemma 2.1(c). This does not hold if G and H have $n-2$ cards in common: the two graphs of order four with two edges have two cards in common but different degree sequences. It is not clear whether this is possible if n is much larger. At present, it is not even known whether, for sufficiently large n, having $n-2$ cards in common is enough to force two graphs of order n to have the same number of edges. Ramachandran and Monikandan [4] proved the following.

Theorem 1.2 *([4]). If G and H are graphs of order $n \geqslant 6$ with minimum degree at least 2 and with $n-2$ cards in common, and $q := |E(H)| - |E(G)| \geqslant 0$, then $q \leqslant 1$.*

Here the condition $n \geqslant 6$ is sharp: see the graphs in Fig. 1(b).

The results of this paper can be summarized as follows.

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http://dx.doi.org/10.1016/j.disc.2015.06.015
0012-365X/

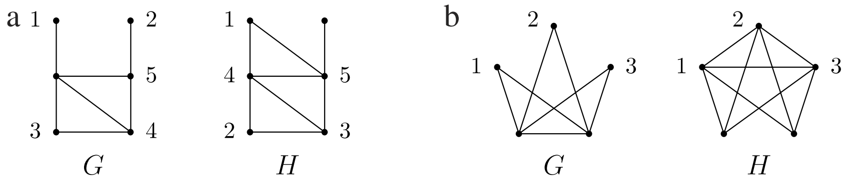

Fig. 1. Examples with (a) $p = q = 1$ and $n = 6$, (b) $p = q = 2$ and $n = 5$.

Theorem 1.3. *Let $p \geqslant 0$ be an integer, let G and H be graphs of order n that have $n - p$ cards in common, and let $q := |E(H)| - |E(G)| \geqslant 0$.*

(a) *If $p \geqslant 1$ and $n \geqslant \max\{5, p^2 + 3\}$, then $q \leqslant p$.*
(b) *If $p \geqslant 2$ and $n \geqslant p^2 + 2p + 2$, then $q \leqslant p - 1$.*
(c) *If $p \geqslant 3$ and $n \geqslant \max\{34, 3p^2 + 1\}$, then $q \leqslant p - 2$.*

It is quite easy to prove that if $n \geqslant p^2 + 2p + 3$ then $q \leqslant p$: see Lemma 2.1(d). Theorem 1.3 is a modest but nontrivial improvement of this.

Theorem 1.1 shows that the result of Theorem 1.3(b) holds if $p = 1$, but only if the condition $n \geqslant p^2 + 2p + 2 = 5$ is strengthened to $n \geqslant 7$. In the case $p = 2$, Theorem 1.3(b) removes the minimum degree condition from Theorem 1.2 at the expense of increasing the lower bound on the order from $n \geqslant 6$ to $n \geqslant 10$. (However, the bound $n \geqslant p^2 + 2p + 2$ is surely not sharp.)

What is missing from Theorem 1.3 is any result showing that $q = 0$, i.e., $|E(G)| = |E(H)|$. It is easy to see that $q = 0$ if $p = 0$ and $n \geqslant 3$, since each edge of G and H appears in $n - 2$ cards. Theorem 1.1 shows that $q = 0$ if $p = 1$ and $n \geqslant 7$. It is not known whether there is any similar result when $p \geqslant 2$.

In Section 2 we make some definitions that we use throughout the paper, and prove some preliminary results, mainly to do with isolated vertices; in particular, Theorem 1.3(a) is proved as Lemma 2.2(d). Some general lemmas about vertices of small degree are proved in Section 3. Theorem 1.3(b) is proved in Section 4, and Theorem 1.3(c) in Section 5.

We need the following simple result, which we prove here since G will have a specific meaning from the start of the next section.

Lemma 1.4. *Let G be a graph of order n with no isolated vertices and containing a vertex w of degree $k \geqslant 2$. Then at most $n-k-1$ cards $G - v$ ($v \in V(G) \setminus \{w\}$) contain an isolated vertex.*

Proof. Suppose $G - v$ contains an isolated vertex x, where $v \in V(G) \setminus \{w\}$. Clearly w and its k neighbors are not isolated in $G - v$, and so x is one of the $n - k - 1$ remaining vertices of G. Also, v is the only neighbor of x in G, and so x is not an isolated vertex of any other graph $G - v'$ ($v' \in V(G) \setminus \{v, w\}$). Thus at most $n - k - 1$ cards $G - v$ ($v \in V(G) \setminus \{w\}$) contain an isolated vertex. $\square$

The degree of a vertex v in a graph G is denoted by $d_G(v)$. A *k-vertex* (*k^--vertex*) is a vertex of degree exactly k (at most k), and $n_k(G)$ ($n_k^-(G)$) denotes the number of k-vertices (k^--vertices) in G. Similarly, a *k-neighbor* (*k^--neighbor*) is a neighbor of degree exactly k (at most k). We occasionally allow k to be negative; naturally $n_k(G) = n_k^-(G) = 0$ if $k < 0$. If G has order n, then 0-vertices and $(n - 1)$-vertices are called *isolated* and *universal*, respectively.

Throughout the paper, all numbers represented by italic letters are integers, and $[a, b]$ denotes the set of integers i such that $a \leqslant i \leqslant b$.

2. Definitions and preliminary results

Throughout the rest of the paper, $p \geqslant 1$, and G and H are graphs of order n that have $n - p$ cards in common, where $|E(H)| - |E(G)| = q \geqslant 0$. The set of common cards is $\mathcal{D} = \{D_1, \ldots, D_{n-p}\}$, and the vertices of G and H are labeled as $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$, respectively, in such a way that $D_i \cong G - v_i \cong H - u_i$ for all $i \in [1, n - p]$. The graphs $D_i \in \mathcal{D}$ should be regarded as disjoint, both from each other and from G and H. Each should be assumed to come equipped with specific isomorphisms $f_i : D_i \to G - v_i$ and $g_i : D_i \to H - u_i$, and each vertex $w \in V(D_i)$ is said to *correspond to* the vertices $f_i(w) \in V(G - v_i)$ and $g_i(w) \in V(H - u_i)$ (and similarly $f_i(w)$ and $g_i(w)$ *correspond to* w).

Let $\Sigma_p := \sum_{i=n-p+1}^{n} (d_G(v_i) - d_H(u_i))$: see Lemma 2.1(b).

Lemma 2.1. (a) *$d_H(u_i) - d_G(v_i) = q$ for all $i \in [1, n - p]$.*
(b) $\Sigma_p = \sum_{i=n-p+1}^{n} (d_G(v_i) - d_H(u_i)) = q(n - p - 2)$.
(c) *If $p = 1$ and $q = 0$ then G and H have the same degree sequence.*
(d) *If $n > p^2 + 2p + 2$ then $q \leqslant p$.*