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On the balanced upper chromatic number of cyclic projective planes and projective spaces

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ABSTRACT

We study vertex colorings of hypergraphs, such that all color sizes differ at most in one (balanced colorings) and each edge contains at least two vertices of the same color (rainbow-free colorings). Given a hypergraph H, the maximum k, such that there is a balanced rainbow-free k-coloring of H is called the balanced upper chromatic number denoted by $\overline{\chi}_b(H)$. Concerning hypergraphs defined by projective spaces, bounds on the balanced upper chromatic number and constructions of rainbow-free colorings are given. For cyclic projective planes of order q we prove that:

$$\frac{q^2+q+1}{6} \leq \overline{\chi}_b(\Pi_q) \leq \frac{q^2+q+1}{3}.$$

We also give bounds for the balanced upper chromatic numbers of the hypergraphs arising from the n-dimensional finite space $\mathrm{PG}(n, q)$.

1. Introduction

The notion of a rainbow-free coloring [11] coincides with the notion of a proper strict coloring of a $\mathcal{C}$-hypergraph in the context of the recent theory of coloring mixed hypergraphs [15]. In this work instead of studying the upper chromatic number our interest is in determining (or estimating) the balanced upper chromatic number, which arises from balanced rainbow-free colorings. We will define all concepts in the general setting of hypergraphs although here we only study projective planes and projective spaces.

Let H be a hypergraph and k be a positive integer. A *k-coloring* of H is a surjective mapping from $V(H)$ to a set of k colors; that is $c : V(H) \to \{0, 1, \dots, k-1\}$. The inverse images of the colors are called the *color classes*; that is $C_i = \{v \in V(H) : c(v) = i\}$ for $i \in \{0, \dots, k-1\}$. Given a k-coloring of H an edge of H is called *rainbow*, if it is totally multicolored (i.e. the coloring assigns pairwise distinct colors to its vertices). A k-coloring of H is said to be *rainbow-free*, if it contains no rainbow edges. The *upper chromatic number* of H, denoted by $\overline{\chi}(H)$, is the largest integer k for which there is a rainbow-free k-coloring of H. We shall observe that, if $\overline{\chi}(H) = k$ then $k+1$ is the minimum integer with the property that every $(k+1)$-coloring of H contains a rainbow edge. In this sense the problem of determining the upper chromatic number is considered an extremal anti-Ramsey problem.

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The upper chromatic number has been studied in many different contexts and has been redefined several times under different names (see [2,3,6,7,10,13,14] and references therein). More specifically, results in projective planes appear for example in [1,4,5]. A natural lower bound for the upper chromatic number is given in terms of the 2-*transversal number* τ_2, which is the minimum cardinality of a set of vertices that intersect each edge in at least two vertices. It is not difficult to see that

$$\overline{\chi}(H) \geq |V(H)| - \tau_2(H) + 1 \tag{1}$$

holds true for every hypergraph H, and it is an interesting problem to determine when this inequality is tight (see [4]).

Generally, the structure of rainbow-free colorings is very specific. Indeed, in order to avoid rainbow edges most of the times very small color classes are needed. For instance, the coloring that provides the above inequality has $|V(H)| - \tau_2(H)$ color classes with one element each and one big color class of order $\tau_2(H)$. Such a coloring is called a *trivial coloring* [4]. The main motivation of this work is to investigate (in contrast to the latter fact), if there are rainbow-free colorings with color classes of almost the same size.

A *balanced coloring* is a coloring in which the cardinality of all color classes differs at most in one. Hence, balanced colorings are in some sense the opposite of trivial colorings. The *balanced upper chromatic number* of H, denoted by $\overline{\chi}_b(H)$, is the largest integer k for which there is a balanced rainbow-free k-coloring of H. Obviously $\overline{\chi}_b(H) \leq \overline{\chi}(H)$ holds true for any hypergraph H. In this work we are interested in showing that the gap between $\overline{\chi}_b(H)$ and $\overline{\chi}(H)$ is large for hypergraphs defined by projective planes. In other words, if we forbid trivial colorings then to avoid rainbow-free colorings we are forced to use much fewer colors. How much fewer is the question that we want to answer.

A projective plane can be considered as a hypergraph where the set of vertices is the set of points, and the edges are the lines. We denote a finite projective plane of order q by Π_q and we will use the same notation to refer to the hypergraph as defined above. So, since Π_q has $v = q^2 + q + 1$ points and the same number of lines each containing $q + 1$ points, then the hypergraph Π_q is a $(q + 1)$-uniform hypergraph of order and size $v = q^2 + q + 1$. It is an easy exercise to verify that the Fano plane Π_2 satisfies $\overline{\chi}(\Pi_2) = 3$ and $\overline{\chi}_b(\Pi_2) = 2$, while the projective plane of order 3, Π_3, satisfies $\overline{\chi}(\Pi_3) = 6$ and $\overline{\chi}_b(\Pi_3) = 4$. For $q = p^h$, being p a large enough prime, it has been recently proved that concerning PG(2, q) the equality holds true in (1), and it is reached only by trivial colorings [4]. In this work we investigate the balanced upper chromatic number $\overline{\chi}_b(\Pi_q)$ of hypergraphs defined by cyclic projective planes.

In Section 2 we recall the notion of cyclic planes. Theoretically, the class of cyclic projective planes is wider than the class of Desarguesian planes, but each known finite cyclic plane is isomorphic to PG(2, q) for a suitable q. We use the polygon model in order to prove that:

$$\frac{q^2+q+1}{6} \leq \overline{\chi}_b(\Pi_q) \leq \frac{q^2+q+1}{3}$$

if Π_q is a cyclic plane of order q. We also show that the upper bound is sharp, if $q \equiv 1 \pmod 3$.

We denote the n-dimensional projective space over the finite field of q elements by PG(n, q). In Section 3 we investigate the hypergraphs arising from PG(n, q), $n \geq 3$. We prove some bounds on the balanced chromatic numbers, and give constructions of new rainbow-free colorings of these hypergraphs.

2. Balanced coloring in cyclic planes

First we give a general upper bound on the balanced upper chromatic number of finite projective planes which is a consequence of an easy counting argument.

Theorem 2.1. *All balanced rainbow-free colorings of any projective plane of order q satisfy that each color class contains at least three points. Thus*

$$\overline{\chi}_b(\Pi_q) \leq \frac{q^2+q+1}{3}.$$

Proof. Suppose to the contrary that the plane has a balanced rainbow-free coloring with a color class of size two. For $i \in \{1, 2, 3\}$, let n_i be the number of color classes of size i. Then $n_2 > 0$, and either $n_1 = 0$ or $n_3 = 0$. Since each point of the plane belongs to exactly one color class, then

$$q^2 + q + 1 = n_1 + 2n_2 + 3n_3. \tag{2}$$

On the other hand, the number of 2-secant lines of a set of one, two, or three points is respectively equal to zero, one, or at most three. Thus the total number of lines which are not rainbow is n_2 if $n_3 = 0$, and at most $n_2 + 3n_3$ if $n_1 = 0$. In both cases, according to (2), we obtain less than $q^2 + q + 1$ not rainbow lines, contradicting that the coloring is rainbow-free. □