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Realizing symmetric set functions as hypergraph cut capacity

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ABSTRACT

A set function on the power set of a finite set is said to be symmetric if the value for each subset coincides with that for its complement. A cut capacity function of an undirected graph or hypergraph is a fundamental example of a symmetric set function, and is also submodular if the capacity on each edge or hyperedge is nonnegative. Fujishige and Patkar (2001) provided necessary and sufficient conditions for set functions to be realized as cut capacity functions of several types of networks.

In this paper, we focus on the case of undirected hypergraphs, which was not dealt with in the previous work of Fujishige and Patkar. For this case, Grishuhin (1989) had given a set of hyperedges forming a basis of the cut realization of symmetric set functions. By using the Möbius inversion formula together with Grishuhin's basis, we extend a result of Fujishige and Patkar for the case of undirected graphs to hypergraphs. We also clarify the kernel of the cut realization, which leads to other interesting bases and an alternative proof of Grishuhin's result. In addition, we provide a new necessary condition for the cut realization by undirected hypergraphs with nonnegative capacity.

1. Introduction

A cut capacity function of a network with nonnegative capacity is a fundamental example of a submodular function. A set function $f : 2^V \to \mathbb{R}$ is said to be *submodular* if it satisfies

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

for every $X, Y \subseteq V$. Submodularity is necessary for set functions to be realized as cut capacity functions of networks with nonnegative capacity, but is far from sufficient. When is a set function realized as a cut capacity function? As an answer to this natural question, Fujishige and Patkar [2] characterized set functions that can be realized as the cut capacity functions of the following types of networks:

(1) a directed graph with a nonnegative capacity on each arc,
(2) an undirected graph with an arbitrary or nonnegative capacity on each edge, and
(3) a directed hypergraph with an arbitrary or nonnegative capacity on each hyperarc, which has exactly one specified tail.

For a finite set V, a set function $f : 2^V \to \mathbb{R}$ is said to be *normalized* if $f(\emptyset) = 0$, and *symmetric* if $f(X) = f(V \setminus X)$ for every $X \subseteq V$. A cut capacity function of a graph or hypergraph is always normalized, and in the undirected case, it is also symmetric regardless of the nonnegativity of the edge (or hyperedge) capacities.

Symmetry of submodular functions is an important property from the viewpoint of optimization. Extending the algorithm of Nagamochi and Ibaraki [10] for finding a minimum cut in an undirected graph with nonnegative capacities,

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Queyranne [13] provided a combinatorial algorithm that finds a proper nonempty subset attaining the minimum value of a symmetric submodular function in $\mathrm{O}(|V|^3\mathrm{EO})$ time, where EO denotes the time to evaluate a function value. It is also well-known that one can minimize an arbitrary submodular function in polynomial time [4–6,9,15]. Applying one of those algorithms repeatedly ($\mathrm{O}(|V|)$ times), one can solve this problem in polynomial time. However, Queyranne's algorithm is much simpler and faster, where it should be remarked that the fastest strongly polynomial-time combinatorial algorithm to minimize a submodular function currently known requires $\mathrm{O}(|V|^5\mathrm{EO} + |V|^6)$ time [12].

The algorithm of Nagamochi and Ibaraki was naturally extended also for finding a minimum cut in an undirected hypergraph by Klimmek and Wagner [7]. Based on this fact, the class of cut capacity functions of undirected hypergraphs with nonnegative capacity is expected to be close to that of symmetric submodular functions.

For the case when each hyperedge can have an arbitrary capacity, Grishuhin [3, Theorem 4.1] provided a set of hyperedges forming a basis of the hypergraph cut realization of symmetric set functions, which implies that any normalized symmetric set function can be realized as a cut capacity function of an undirected hypergraph. Such a realization is not unique in general, since the number of possible hyperedges in a hypergraph with the vertex set V is $2^{|V|}$, which means that the degree of freedom of the cut realization is $2^{|V|}$, whereas the degree of freedom of a normalized symmetric set function $f : 2^V \to \mathbb{R}$ is $2^{|V|-1} - 1$.

To fill this gap, we introduce various types of "standard forms" of hypergraphs, whose hyperedge sets are bases of cut realization (see Section 3.2 for the details). While one of them is equivalent to [3, Theorem 4.1], we give an alternative proof by showing explicitly the kernel of cut realization (see Section 4.2). Moreover, using the Möbius inversion formula together with the standard forms, we extend a result of Fujishige and Patkar [2] for undirected graphs to undirected hypergraphs with the size of hyperedges bounded (Theorem 3.2), and present a new necessary condition for the cut realization by undirected hypergraphs with nonnegative capacity (Theorem 3.3).

The rest of this paper is organized as follows. In Section 2, we give necessary definitions and describe related results due to Fujishige and Patkar [2] and Grishuhin [3]. Section 3 is devoted to presenting our results: the realizability of symmetric set functions as cut capacity functions of undirected hypergraphs, and various standard forms for such realization. The proofs of these results are provided in Section 4, which also presents some interesting corollaries on symmetric set functions.

2. Preliminaries

2.1. Set functions

A *set function* is a function defined on a set family. Throughout this paper, we consider only set functions defined on families of subsets of a finite set V, which is called the *ground set*. We say that a set function $f : 2^V \to \mathbb{R}$ is *normalized* if $f(\emptyset) = 0$, and *symmetric* if $f(X) = f(V \setminus X)$ for every $X \subseteq V$.

For a finite set V and $i = 0, 1, \ldots, |V|$, let $\binom{V}{i}$ denote the family of i-element subsets of V, i.e., $\binom{V}{i} = \{X \subseteq V \mid |X| = i\}$. The following two propositions are special cases of the Möbius inversion formula (see, e.g., [1,14]).

Proposition 2.1. *For a set function $f : 2^V \to \mathbb{R}$, there exists a unique family of set functions $f^{(i)} : \binom{V}{i} \to \mathbb{R}$ $(i = 0, 1, \ldots, |V|)$ such that*

$$f(X) = \sum_{Y \subseteq X} f^{(|Y|)}(Y)$$

for every $X \subseteq V$. Moreover, they are explicitly written as

$$f^{(i)}(X) = \sum_{Y \subseteq X} (-1)^{|X \setminus Y|} f(Y)$$

for each $i = 0, 1, \ldots, |V|$ and each $X \in \binom{V}{i}$.

Proposition 2.2. *For a symmetric set function $f : 2^V \to \mathbb{R}$ and a fixed element $r \in V$, there exists a unique family of set functions $f_r^{(i)} : \{X \mid r \in X \in \binom{V}{i}\} \to \mathbb{R}$ $(i = 1, 2, \ldots, |V|)$ such that*

$$f(X) = \sum_{Y :\, r \in Y \subseteq X} f_r^{(|Y|)}(Y)$$

for every $X \subseteq V$ with $r \in X$. Moreover, they are explicitly written as

$$f_r^{(i)}(X) = \sum_{Y :\, r \in Y \subseteq X} (-1)^{|X \setminus Y|} f(Y) = f^{(i)}(X) + f^{(i-1)}(X - r)$$

for each $i = 1, 2, \ldots, |V|$ and each $X \in \binom{V}{i}$ with $r \in X$.