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A list version of graph packing

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ABSTRACT

We consider the following generalization of graph packing. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order n and $G_3 = (V_1 \cup V_2, E_3)$ a bipartite graph. A bijection f from V_1 onto V_2 is a *list packing* of the triple (G_1, G_2, G_3) if $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and for all $v \in V_1$, $vf(v) \notin E_3$. We extend the classical results of Sauer and Spencer and Bollobás and Eldridge on packing of graphs with small sizes or maximum degrees to the setting of list packing. In particular, we extend the well-known Bollobás–Eldridge Theorem, proving that if $\Delta(G_1) \le n-2$, $\Delta(G_2) \le n-2$, $\Delta(G_3) \le n-1$, and $|E_1| + |E_2| + |E_3| \le 2n-3$, then either (G_1, G_2, G_3) packs or is one of 7 possible exceptions.

1. Introduction

The notion of graph packing is a well-known concept in graph theory and combinatorics. Two graphs on n vertices are said to *pack* if there is an edge-disjoint placement of the graphs onto the same set of vertices. In 1978, two seminal papers, [6] and [1], on extremal problems on graph packing appeared in the same journal. In particular, Sauer and Spencer [6] proved sufficient conditions for packing two graphs with bounded product of maximum degrees.

Theorem 1 ([6]). *Let G_1 and G_2 be two graphs of order n. If $2\Delta(G_1)\Delta(G_2) < n$, then G_1 and G_2 pack.*

This result is sharp and later Kaul and Kostochka [5] characterized all graphs for which Theorem 1 is sharp.

Theorem 2 ([5]). *Let G_1 and G_2 be two graphs of order n and $2\Delta(G_1)\Delta(G_2) \le n$. Then G_1 and G_2 do not pack if and only if one of G_1 and G_2 is a perfect matching and the other is either $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.*

Bollobás and Eldridge [1] and, independently, Sauer and Spencer gave sufficient conditions for packing two graphs with given total number of edges.

Theorem 3 ([1,6]). *Let G_1 and G_2 be two graphs of order n. If $|E(G_1)| + |E(G_2)| \le \frac{3}{2}n - 2$, then G_1 and G_2 pack.*

This result is best possible, since $G_1 = K_{1,n-1}$ and $G_2 = \frac{n}{2}K_2$ do not pack. Bollobás and Eldridge [1] proved the stronger result that the bound of Theorem 3 can be significantly strengthened when $\Delta(G_1) < n-1$ and $\Delta(G_2) < n-1$.

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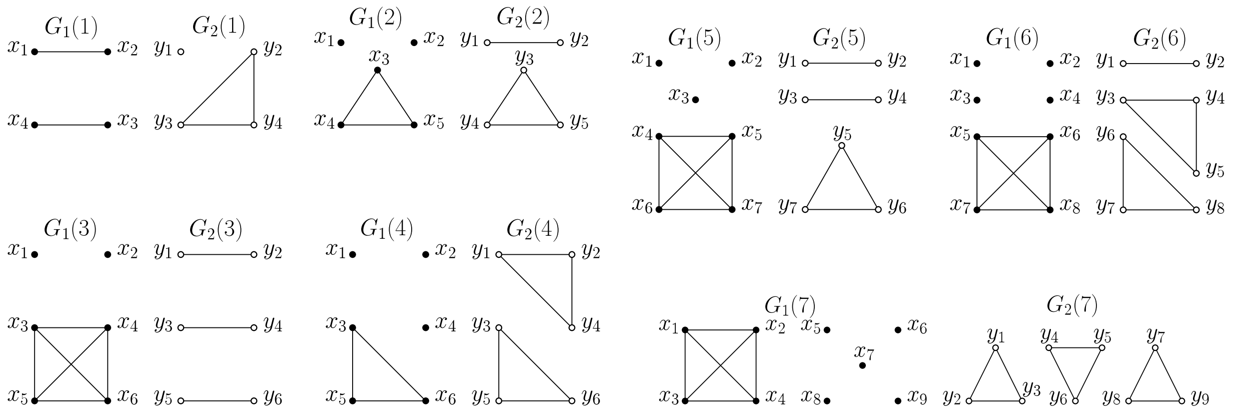

Fig. 1. Bad pairs in Theorems 4 and 6.

Theorem 4 ([1]). *Let G_1 and G_2 be two graphs of order n. If $\Delta(G_1), \Delta(G_2) \leq n-2$, $|E(G_1)| + |E(G_2)| \leq 2n-3$, and $\{G_1, G_2\}$ is not one of the following pairs: $\{2K_2, K_1 \cup K_3\}$, $\{\overline{K}_2 \cup K_3, K_2 \cup K_3\}$, $\{3K_2, \overline{K}_2 \cup K_4\}$, $\{\overline{K}_3 \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K}_3 \cup K_4\}$, $\{\overline{K}_4 \cup K_4, K_2 \cup 2K_3\}$, $\{\overline{K}_5 \cup K_4, 3K_3\}$ (Fig. 1), then G_1 and G_2 pack.*

This result is also sharp, since the graphs $G_1 = C_n$ and $G_2 = K_{1,n-2} \cup K_1$ satisfy the maximum degree conditions, have $2n-2$ edges, and do not pack. There are other extremal examples.

Variants of the packing problem have been studied and, in particular, restrictions of permissible packings arise both within proofs and are posed as independent questions. The notion of a bipartite packing was introduced by Catlin [2] and was later studied by Hajnal and Szegedy [4]. This variation of traditional packing involves two bipartite graphs $G_1 = (X_1 \cup Y_1, E_1)$ and $G_2 = (X_2 \cup Y_2, E_2)$ where permissible packings send X_1 onto X_2 and Y_1 onto Y_2. The problem of fixed-point-free embeddings, studied by Schuster in 1978, considers a different restriction to the original packing problem [7]. In this case, $G_1 = G$ is packed with $G_2 = G$ under the additional restraint that no vertex of G_1 is mapped to its copy in G_2. In [9], Schuster's result is used to prove a necessary condition for packing two graphs with given maximum and average degrees.

In this paper, we introduce the language of list packing in order to model such problems. A *list packing* of the graph triple (G_1, G_2, G_3) with $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_1 \cup V_2, E_3)$ is a bijection $f : V_1 \to V_2$ such that $uv \in E_1$ implies $f(u)f(v) \notin E_2$ and for each $u \in V_1$, $uf(u) \notin E_3$. Note that both G_1 and G_2 are graphs on n vertices so that G_3 has $2n$ vertices, and one can think of the edge set E_3 as a list of restrictions that must be avoided when packing G_1 and G_2.

This notion is closely related to Vizing's concept of list coloring [8]. Suppose we wish to color a graph G with the colors $\{1, \ldots, k\}$. A *list assignment* L is a function on the vertex set $V(G)$ that returns a set of colors $L(v) \subseteq \{1, \ldots, k\}$ permissible for v. A *list coloring*, more specifically an *L-coloring*, is a proper coloring f of G such that $f(v) \in L(v)$ for all $v \in V(G)$. The problem of list coloring G can be stated within the framework of list packing. A proper L-coloring of a graph G is equivalent to a list packing where $G_1 = G$ along with an appropriate number of isolated vertices, G_2 is a disjoint union of K_n's each representing a color, and E_3 consists of all edges going between a vertex $v \in V_1$ and the copies of K_n corresponding to colors *not* in $L(v)$. Note the list $L(v)$ denotes permissible colors in a list coloring while $N_3(v)$ specifies forbidden vertices in a list packing.

Similarly, the variations of packing discussed above can be modeled using this framework. A bipartite packing is a packing of the triple (G_1, G_2, G_3) where E_3 consists of all edges between X_i and Y_{3-i} for $i = 1, 2$. A fixed-point-free embedding is a packing of the triple (G, G, G_3) where $E_3 = \{(v, v) : v \in V(G)\}$. Further, several important theorems on the ordinary packing can be stated in terms of list packing. The results of this paper prove natural generalizations of Theorems 1–4. In particular, we extend Theorems 1 and 2 as follows.

Theorem 5. *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If*

$$\Delta(G_1)\Delta(G_2) + \Delta(G_3) \leq n/2, \tag{1}$$

then $\mathbf{G}$ does not pack if and only if $\Delta(G_3) = 0$ and one of G_1 or G_2 is a perfect matching and the other is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$. Consequently, if $\Delta(G_1)\Delta(G_2) + \Delta(G_3) < n/2$, then $\mathbf{G}$ packs.

The main result of this paper is the following list version of Theorem 4.

Theorem 6. *Let $\mathbf{G} = (G_1, G_2, G_3)$ be a graph triple with $|V_1| = |V_2| = n$. If $\Delta(G_1), \Delta(G_2) \leq n-2$, $\Delta(G_3) \leq n-1$, $|E_1| + |E_2| + |E_3| \leq 2n-3$ and the pair $\{G_1, G_2\}$ is none of the 7 pairs in Fig. 1, then $\mathbf{G}$ packs.*

Theorem 6 is sharp and has more sharpness examples than Theorem 4. First, the condition $\Delta(G_3) \leq n-1$ cannot be removed, since a vertex in V_1 adjacent to all vertices in V_2 cannot be placed at all (Fig. 2(a)). The restriction on $|E_1|+|E_2|+|E_3|$ is also sharp, as there are several examples of graphs with $|E_3| > 0$ and edge sum equal to $2n-2$ that do not pack.