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Note

Threefold triple systems with nonsingular N_2

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ABSTRACT

There are various results connecting ranks of incidence matrices of graphs and hypergraphs with their combinatorial structure. Here, we consider the generalized incidence matrix N_2 (defined by inclusion of pairs in edges) for one natural class of hypergraphs: the triple systems with index three. Such systems with nonsingular N_2 (over the rationals) appear to be quite rare, yet they can be constructed with PBD closure. In fact, a range of ranks near $\binom{v}{2}$ is obtained for large orders v.

1. Introduction

We consider hypergraphs with the possibility of repeated edges. Let v and λ be positive integers, and suppose $K \subset \mathbb{Z}_{\geq 2} := \{2, 3, 4, \ldots\}$. A *pairwise balanced design* $\mathrm{PBD}_\lambda(v, K)$ is a hypergraph $(V, \mathcal{B})$ with v vertices, edge sizes belonging to K, and such that

- any two distinct vertices in V appear together in exactly λ edges.

In this context, vertices are also called *points* and edges are normally called *blocks*. The parameter λ is the *index*; often it is taken to be 1 and suppressed from the notation. We remark that K could contain unused block sizes.

There are numerical constraints on v given λ and K. An easy double-counting argument on pairs of points leads to the *global condition*

$$\lambda v(v-1) \equiv 0 \pmod{\beta(K)}, \tag{1.1}$$

where $\beta(K) := \gcd\{k(k-1) : k \in K\}$. Similarly, counting incidences with any specific point leads to the *local condition*

$$\lambda(v-1) \equiv 0 \pmod{\alpha(K)}, \tag{1.2}$$

where $\alpha(K) := \gcd\{k-1 : k \in K\}$. Wilson's theory, [9], asserts that (1.1) and (1.2) are sufficient for large v.

In the case $K = \{3\}$, we obtain a (λ-fold) *triple system* or $\mathrm{TS}_\lambda(v)$. When $\lambda = 1$ we have a *Steiner triple system* and it is well known that these exist for all $v \equiv 1, 3 \pmod 6$. In this article we are especially interested in the case $\lambda = 3$. The divisibility conditions (1.1) and (1.2) simply reduce to v being odd. There are $3v(v-1)/6 = \binom{v}{2}$ blocks. For a comprehensive reference on triple systems, the reader is referred to Colbourn and Rosa's book [4].

Given any hypergraph $H = (V, E)$, we may define its *incidence matrix* $N = N(H)$ as the zero–one inclusion matrix of points versus edges. That is, N has rows indexed by V, columns indexed by E, and where, for $x \in V$, $e \in E$,

$$N(x, e) = \begin{cases} 1 & \text{if } x \in e; \\ 0 & \text{otherwise.} \end{cases}$$

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Linear algebraic properties of incidence matrices have received a lot of attention. Especially interesting are connections with the underlying combinatorial structure. We give two classical examples. First, in the case of ordinary graphs, in which $E \subseteq \binom{V}{2}$, it is known [8] that N has full rank (over $\mathbb{R}$) if and only if every connected component is non-bipartite. As a different example, the rank of a Steiner triple system over the binary field $\mathbb{F}_2$ is connected in [5] with its 'projective dimension'. This measures the length of the lattice of largest possible proper subsystems.

Let s be a positive integer. The *higher incidence matrix* N_s has a similar definition, but where rows are indexed by $\binom{V}{s}$ (the s-subsets of vertices), columns are again indexed by blocks, and entries are defined by inclusion. That is, for $S \subseteq V$, $|S| = s$, and $e \in E$, we have

$$N_s(S, e) = \begin{cases} 1 & \text{if } S \subseteq e; \\ 0 & \text{otherwise.} \end{cases}$$

Higher incidence matrices were used by Ray-Chaudhuri and Wilson in [7] to extend Fisher's inequality to designs of 'higher strength'. In a little more detail, suppose we have a system $(V, \mathcal{B})$ of v points, blocks of a fixed size k, and every t-subset of points belongs to exactly λ blocks. These are sometimes denoted $S_\lambda(t, k, v)$. Suppose further that t is even, say $t = 2s$, and $v \geq k + s$. Then the conclusion is that $|\mathcal{B}| \geq \binom{v}{s}$, and it comes with a strong structural condition for equality. The matrix N_s plays a key role in the proof. Incidentally, a new result of Keevash in [6] proves that, for large v, the divisibility conditions $\binom{k-i}{t-i} \mid \lambda\binom{v-i}{t-i}$ for $i = 0, \ldots, t$ (which are the analogs of (1.1)–(1.2)) suffice for the existence of $S_\lambda(t, k, v)$.

Returning to pairwise balanced designs, higher incidence matrices are of limited use when $\lambda = 1$. In this case, the matrix N_2 is only slightly interesting; each of its rows has exactly one nonzero entry. The matrix N_k is just, under a reordering of rows, the identity matrix on top of the zero matrix. In between, N_s for $2 < s < k$ has many zero rows and not much structure.

We would like to consider N_2 for what is perhaps the first natural case: threefold triple systems $TS_3(v)$. For such designs, N_2 is square of order $\binom{v}{2}$. In general, we observe that the property of a design having full rank N_2 is 'PBD-closed'. From this and some small designs, we have the following main result.

Theorem 1.1. *There exists a* $TS_3(v)$ *with* N_2 *nonsingular over* $\mathbb{R}$ *for all odd* $v \geq 5$ *except possibly for* $v \in E_{579} := \{v : v \equiv 1 \pmod 2, v \geq 5, \text{ and } \nexists\, PBD(v, \{5, 7, 9\})\}$.

It is known (see [1] and the summary table entry at [2], page 252) that

$$E_{579} \subseteq \{11..19, 23, 27..33, 39, 43, 51, 59, 71, 75, 83, 87, 95, 99, 107, 111, 113, 115, 119, 139, 179\},$$

and therefore Theorem 1.1 settles the existence question for all but a finite set of values v.

The next section sets up and completes the proof. Then, we conclude with a short discussion of some related topics, including a brief look at such ranks in characteristic p.

2. PBD closure and proof of the main result

To prove Theorem 1.1, we first observe that having square nonsingular N_2 is a 'PBD-closed' property.

Lemma 2.1. *Suppose there exists a* $PBD(v, L)$ *and, for each* $u \in L$*, there exists a* $PBD_\lambda(u, K)$ *having* N_2 *square and full rank over* $\mathbb{F}$*. Then there exists a* $PBD_\lambda(v, K)$ *having* N_2 *square and full rank over* $\mathbb{F}$*.*

Proof. Suppose our $PBD(v, L)$ is $(V, \mathcal{A})$. Construct a $PBD_\lambda(v, K)$ with points V and block collection

$$\mathcal{B} = \bigcup_{U \in \mathcal{A}} \mathcal{B}[U], \tag{2.1}$$

where $\mathcal{B}[U]$ denotes the blocks of a $PBD_\lambda(|U|, K)$ on U having full rank N_2. (Note (2.1) should be interpreted as a formal sum or 'multiset union'.) It is clear that $(V, \mathcal{B})$ is a $PBD_\lambda(v, K)$. Consider its incidence matrix $N_2(\mathcal{B})$. If columns are ordered respecting some ordering $U_1, U_2, \ldots$ of $\mathcal{A}$ and the union in (2.1), and rows are ordered respecting $\binom{U_1}{2}, \binom{U_2}{2}, \ldots$, then we obtain a block-diagonal structure

$$N_2(\mathcal{B}) = N_2(\mathcal{B}[U_1]) \oplus N_2(\mathcal{B}[U_1]) \oplus \cdots.$$

Since each block is nonsingular, so is $N_2(\mathcal{B})$. □

To clarify, we are working in characteristic zero (rank computed over $\mathbb{Q}$) throughout the remainder of the section.

Lemma 2.2. *For* $v = 5, 7, 9$*, there exists a* $TS_\lambda(v)$ *having nonsingular* N_2*.*

Proof. The unique $TS_3(5)$ is just the complete design $\binom{[5]}{3}$. Accordingly, for this design, we have $N_2 N_2^\top = 3I + A$, where A is the adjacency matrix of the line graph of K_5 (or complement of the Petersen graph). Since A is known to have eigenvalues $(-2)^5, 1^4, 6^1$, it follows that N_2 has full rank. Examples for $v = 7, 9$ are given below as a list of blocks on $\{0, \ldots, v-1\}$.

$v = 7$:

{0, 1, 2}, {0, 1, 3}, {0, 1, 4}, {0, 2, 3}, {0, 2, 5}, {0, 3, 6}, {0, 4, 5},
{0, 4, 6}, {0, 5, 6}, {1, 2, 4}, {1, 2, 6}, {1, 3, 5}, {1, 3, 6}, {1, 4, 5},
{1, 5, 6}, {2, 3, 4}, {2, 3, 5}, {2, 4, 6}, {2, 5, 6}, {3, 4, 5}, {3, 4, 6}.