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The covering radius problem for sets of 1-factors of the complete uniform hypergraphs

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ABSTRACT

Covering and packing problems constitute a class of questions concerning finite metric spaces that have surfaced in recent literature. In this paper, we consider, for the first time, these problems for the finite metric space (Ω, d) arising from the set Ω of 1-factors of the complete t-uniform hypergraph $\mathcal{H}$ on nt vertices for some positive integers n and t. We focus on the covering problem; in particular we investigate bounds on the covering radius of any code $C \subseteq \Omega$. In doing so, we give both upper and lower bounds on the covering radius, as well as a frequency parameter type result that follows from the Lovász local lemma.

1. Introduction

Covering and packing problems (see [5] for a historic and comprehensive survey) first arose in the theory of error-correcting codes, whereby researchers, following the Hamming approach towards engineering good codes, sought to construct codes whose *packing radius* and *covering radius* are the same, which are also known as perfect codes. Recall that, for a code $C \subseteq \Omega$ where Ω is the underlying space with metric d, its packing radius $pr(C)$ is the maximum radius $r_{\max}$ such that balls of radius $r_{\max}$ centered on its codewords are pairwise disjoint; its covering radius $cr(C)$ is the minimum radius $r_{\min}$ such that balls of radius $r_{\min}$ centered on its codewords cover the entire underlying space Ω.

While many fundamental and practical problems in coding are concerned with linear or block codes over finite fields, questions about covering and packing radius can be asked for any subset C of arbitrary finite metric spaces. For instance, the covering radius problem for subsets of the symmetric group S_n endowed with the Hamming metric (see Cameron–Wanless [4] and Ku–Keevash [9]) was motivated by practical problems arising from permutation array codes, which reportedly have applications in powerline communication models [6].

This paper explores, for the first time, covering problems for the finite metric space arising from the set Ω of 1-factors of the complete t-uniform hypergraph, which we denote by $\mathcal{H}$. Remark that $\mathcal{H}$ refers to that hypergraph such that $|V| = tn$ for natural numbers n and t, and the edge set of $\mathcal{H}$ is the set $\{E_1, \ldots, E_{\binom{nt}{t}}\}$ of all t-edges of V. Accordingly, a 1-factor M of $\mathcal{H}$ is a set of n pairwise disjoint t-edges of $\mathcal{H}$ whose union is V. Our metric is motivated by the Hamming metric, and is defined as follows. Given any two 1-factors M and M', we have

$$d(M, M') = n - |M \cap M'|.$$

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It is clear that such a metric is of Hamming type as it measures the number of edges in which the two matchings differ, just as how the Hamming distance between any two block codes measures the number of positions in which their letters differ.

Notwithstanding the general scope of our investigation, we focus mainly on the covering problem. The remaining sections of this paper are organized as follows. First, we introduce the coding-theoretic concepts as they are adapted to fit our investigation. We then establish a Hamming bound-type result which gives us information about the packing and covering radii of a code of Ω. Having derived some bounds on the covering radius of a code using simple probabilistic methods, we proceed to prove our main theorem (Theorem 3.5) via the Lovász local lemma. In conclusions, we present some numerical results computed based on the various bounds established, and discuss several possible extensions of the problems as future avenues for further research.

2. Codes on 1-factors

In the theory of error correction codes, researchers who study block codes are interested in three so-called fundamental code parameters: size, minimal distance, and length, which refer, respectively, to (i) the cardinality m of the set of codewords; (ii) the minimum distance $d_{\min}$ between any two different codewords; and (iii) the length n of each codeword. They seek to show the existence, construction, as well as classification of codes achieving extremal values in one parameter upon fixing the other two parameters. The interested reader should consult [7] for further details.

Now, consider the metric space (Ω, d) of all 1-factors of the complete t-uniform hypergraph $\mathcal{H}$ on tn vertices, whereby d is defined by $d(M, M') = n - |M \cap M'|$ for any two 1-factors M and M'. It can be easily shown that $|\Omega| = \frac{(tn)!}{n!\cdot(t!)^n}$. Call any subset $\mathcal{M} \subseteq \Omega$ a code of Ω. It is not too difficult to verify that d satisfies the axioms of a metric space. Each 1-factor $M \in \mathcal{M}$ is called a codeword. For a codeword M contained in a code $\mathcal{M}$, let the ball $B_r(M)$ of radius r around M be the set of all 1-factors $M' \in \Omega$ such that $d(M, M') \le r$. The cardinality $|B_r(M)|$ of this ball is called its volume. It is apparent (by symmetry) that $|B_r(M)| = |B_r(M')|$ for any $M \neq M'$; thus we will just write $|B_r|$ to denote the volume of a ball of radius r.

Remark that for a code $\mathcal{M} \subseteq \Omega$, we have

$$pr(\mathcal{M}) := \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor;$$
$$cr(\mathcal{M}) := \max_{M' \in \Omega} \min_{M \in \mathcal{M}} d(M, M').$$

It turns out that there is a close relationship between the packing and covering radii of codes and the fundamental code parameters. For a code $\mathcal{M}$ being communicated under the nearest neighbor decoding scheme and with minimum distance $d_{\min}$, we can guarantee that any codeword M in $B_{\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor}$ will be decoded correctly. In other words, a code with minimum distance $d_{\min}$ can correct up to $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor$ errors. It follows, by the definitions of the packing radius and the covering radius, that $\left\lfloor \frac{d_{\min}-1}{2} \right\rfloor = pr(\mathcal{M}) \le cr(\mathcal{M})$. Thus, it is clear that determining the covering radius of a code $\mathcal{M}$ provides an upper bound to its packing radius, while determining the packing radius provides a lower bound to the covering radius.

Below, we give a simple inequality, which is an analogue of the well-known Hamming sphere-packing bound, that provides a lower bound for the covering radius.

2.1. A classical bound

Let $\mathcal{M}$ be a code of Ω with minimum distance d. Using nearest neighbor decoding, the code can correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors; that is, balls $B_{\left\lfloor \frac{d-1}{2} \right\rfloor}$ of radius $\left\lfloor \frac{d-1}{2} \right\rfloor$ centered on the codewords do not intersect. This implies that the union of the balls centered on the codewords at most covers the entire metric space. In other words, the following inequality must hold.

Theorem 2.1 (*Hamming Sphere-Packing Bound*)**.** *For a given code $\mathcal{M} \subseteq \Omega$ with minimum distance d,*

$$|\mathcal{M}| \cdot \left| B_{\left\lfloor \frac{d-1}{2} \right\rfloor} \right| \le |\Omega|. \tag{1}$$

Observe that Theorem 2.1 implies the following inequality.

$$pr(\mathcal{M}) = \left\lfloor \frac{d-1}{2} \right\rfloor \le cr(\mathcal{M}). \tag{2}$$

In particular, if the inequality in (1) is strict, then so is the inequality between $cr(\mathcal{M})$ and $\left\lfloor \frac{d-1}{2} \right\rfloor$ in (2).