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Axiomatic characterization of the interval function of a block graph

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ABSTRACT

In 1952 Sholander formulated an axiomatic characterization of the interval function of a tree with a partial proof. In 2011 Chvátal et al. gave a completion of this proof. In this paper we present a characterization of the interval function of a block graph using axioms on an arbitrary transit function R. From this we deduce two new characterizations of the interval function of a tree.

1. Introduction

One of the fundamental notions of metric graph theory is that of the *interval function* $I : V \times V \to 2^V$ of a connected graph $G = (V, E)$, where $I(u, v)$ is the set of vertices on shortest paths between u and v in G. The term interval function was coined in [16], which is the first extensive study of this function. The notion already existed long before. We do not know for sure, but one of the first occurrences might be the thesis of W.D. Duthie [11] of 1940 on "Segments in Ordered Sets", see also [12]. He characterized distributive lattices by postulates or 'axioms' on segments. This work was pursued by Sholander [26,27] in the early 1950s. Sholander studied median semilattices using segments. Median semilattices can also be studied as graphs: the Hasse diagram of a median semilattice is precisely a median graph (and vice versa). This was done for the first time by Avann in 1963 (who called the graph a 'unique ternary distance graph'), and later independently by Nebeský in 1971, and

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Mulder [15,19,16] in 1978–1980. Sholander also presented a set of axioms on segments that characterizes the segments (intervals) of a tree. But he gave a partial proof for this characterization. Only recently, in 2011, a completion of this proof was presented by Chvátal, Rautenbach and Schäfer [10].

Sholander also pursued another line of study in his papers [26,27], viz. that of betweenness in the language of ternary relations. This generalized results by Pitcher and Smiley [25]. Sholander used this notion of betweenness to characterize median betweenness, a structure that is equivalent to median semilattices and median graphs, see e.g. [20,16,21]. Amongst the results in [26] was a characterization of a tree betweenness. A new characterization of a tree betweenness was obtained recently by Burigana [2], with a short new proof by Chvátal et al. [10].

The focus of Sholander was on sets of axioms with as few axioms as possible. This was also the approach of later authors, see [20,10]. In this approach the axioms are necessarily of rather complex nature. In [16] and later work a different approach was taken: here the choice has been to find axioms that are as elementary as possible, and also such that they are applicable in the most general setting, not that of only very well-structured graphs or ordered sets (such as median graphs and median semilattices), see e.g. [22–24,6]. In [16] five simple and elementary properties of the interval function were given that are now known as the 'five classical' axioms for the interval function. In [18] the interval function of a connected graph is characterized by a set of axioms that includes these five classical, elementary axioms. The approach in [18] was as follows. First as much as possible was deduced using the five classical axioms only. Then the road blocks were determined that prevented any further consequence. From these road blocks two more axioms were inferred that, together with the five classical axioms, characterize the interval function of a connected graph. These two extra axioms were more complicated, but still minimal in the sense that weaker axioms would not do the trick.

In [14] a similar approach for betweenness was chosen: using axioms as simple as possible to study betweenness in a broad context. As opposed to the above idea of betweenness as a ternary relation, a betweenness in [14] was formulated in terms of a function $R : V \times V \to 2^V$. One advantage of this approach is that now it could be used in other contexts. In [17] a unifying approach for moving around in discrete structures such as graphs and partially ordered sets was presented: a *transit function* $R : V \times V \to 2^V$ satisfying three elementary axioms. It includes all of the above functions, but also other so-called path functions, like the *induced path function* J, see [6,7], where $J(u, v)$ consists of the vertices on induced paths between u and v. Recently in [4] characterizations of some graph classes were obtained using betweenness axioms on the interval function and on the induced path function.

In this paper we return to the interval function. Above we mentioned the Sholander characterization (with a completion of the proof by Chvátal et al.) by a set of axioms with as few axioms as possible. Here we choose the other approach (from [16,14,18]): try to find a set of axioms that are each as simple and elementary as possible. We present a characterization of the interval function of a block graph. All but one of the axioms are simple and elementary in the sense that these are the axioms for a betweenness from [14]. As corollaries we obtain two new characterizations in the case of trees. Our sets of axioms have one axiom in common with the Sholander set for trees. In our characterizations the remaining axioms form an actually weaker set than those in the Sholander characterization.

A betweenness sensu Sholander is a special type of a ternary relation $\mathcal{B}$ on V, where a triple (u, x, v) is in $\mathcal{B}$ means that x is *between* u and v. We can translate this into a function $R : V \times V \to 2^V$ by defining $R(u, v)$ to be the set of all x between u and v. The axioms on $\mathcal{B}$ then translate to axioms on R. With this translation in mind we study the tree betweenness of Burigana [2] and Chvátal et al. [10]. Below we obtain another characterization of the interval function of a tree that involves axioms that are actually weaker than the axioms of Sholander and those of Burigana/Chvátal et al. For more information on tree betweenness and other literature we refer the reader to their papers [26,2,10].

We investigate the independence of the axioms in our various characterizations. We present our results in the context of transit functions. Besides this we present a characterization of the interval function of a path and a star.

2. Axioms on transit functions

Throughout this paper V is a finite nonempty set. A *transit function* on V is a function $R : V \times V \to 2^V$, where 2^V is the power set of V, satisfying the following three axioms.

($t1$) $u \in R(u, v)$, for all u, v in V.
($t2$) $R(u, v) = R(v, u)$, for all u, v in V.
($t3$) $R(u, u) = \{u\}$, for all u in V.

The third axiom could be deleted. It is usually added to exclude degenerate cases. For instance, the function $F(u, v) = V$, for all u, v in V, satisfies the first two axioms, but will not enlighten us about any aspect of an underlying structure. In the sequel we will see that in many relevant cases ($t3$) follows from other axioms. If $G = (V, E)$ is a graph with vertex set V, then we say that R is a transit function on G. The *underlying graph* G_R of a transit function R is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u, v) = \{u, v\}$. Note that in general G and G_R need not be isomorphic graphs, see [17]. For one instance of this phenomenon we refer to Example 18.

The notion of transit function was introduced in [17] as a unifying concept for many functions on graphs that have been studied so far, e.g. the (geodesic) interval function I, the induced path function J, see [6,7], the triangle-path function T, see [5,9], the all-paths function A, see [3]. It was also meant to create a framework for new problems and ideas. The four mentioned functions are all so-called *path transit functions*, because they are defined in terms of paths in G. See [17,8] for