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Regular pseudo-oriented maps and hypermaps of low genus

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ABSTRACT

Pseudo-orientable maps were introduced by Wilson in 1976 to describe non-orientable regular maps for which it is possible to assign an orientation to each vertex in such a way that adjacent vertices have opposite orientations. This property extends naturally to non-orientable and orientable hypermaps. In this paper we classify the regular pseudo-oriented maps and hypermaps of characteristic $\chi \geqslant -3$. With the help of GAP (The GAP group, 2014) and its library of small groups, we extend the classification down to characteristic $\chi = -16$ (Tables 7–19 in the Appendix).

1. Introduction

Pseudo-orientable maps were introduced by Wilson [13] in 1976 to describe non-orientable regular maps for which it is possible to assign an orientation to each vertex in such a way that adjacent vertices have opposite orientations. This property was extended and translated into group theoretical language by Breda [1] as a particular case of Θ-marked hypermaps. More precisely, pseudo-orientable hypermaps correspond to conjugacy classes of subgroups of finite index in Δ^0, the index 2 subgroup of the free product

$$\Delta := \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle R_0, R_1, R_2 \mid (R_0)^2, (R_1)^2, (R_2)^2 \rangle$$

generated by $A := R_0$, $B := R_0^{R_1}$ and $Z := R_1R_2$. Normal subgroups H of finite index in Δ^0 correspond to *regular* pseudo-orientable hypermaps. These are described by the four-tuple $(G; a, b, z)$ consisting of the finite group $G := \Delta^0/H$ generated by $a := HA$, $b := HB$ and $z := HZ$. Topological interpretation of these objects as particular triangulations of compact and connected surfaces allows a classification by Euler characteristic.

The paper is organized as follows. In Section 2 we give some preliminaries going from the combinatorial definition of hypermaps to the algebraic characterization, passing through the topological representation as triangulations of surfaces. In Section 3 we characterize regular pseudo-oriented hypermaps and give some examples. In Section 4 we classify regular pseudo-oriented hypermaps of characteristic $\chi \geqslant -3$ and in the Appendix we extend this classification down to characteristic $\chi = -16$ with the help of GAP [11] and its library of small groups.

Throughout the paper actions and functions are right-handed, that is, they act on the right.

2. Preliminaries

A (*boundary free*) *hypermap* $\mathcal{H}$ is a four-tuple $(F; r_0, r_1, r_2)$, where F is a finite non-empty set of *flags*, and r_0, r_1, r_2 are fixed-point-free involutions of F generating a transitive (permutation) group $Mon(\mathcal{H}) := \langle r_0, r_1, r_2 \rangle$, called the *monodromy*

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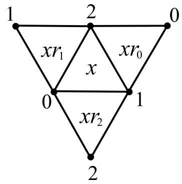

Fig. 1. Gluing of triangles.

group of $\mathcal{H}$. A hypermap $(F; r_0, r_1, r_2)$ satisfying $(r_0r_2)^2 = 1$ is called a *map*. Given a hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ we set

$$M_0 := \langle r_1, r_2\rangle, \qquad M_1 := \langle r_2, r_0\rangle \quad \text{and} \quad M_2 := \langle r_0, r_1\rangle.$$

Then M_0, M_1 and M_2 are dihedral groups of order $2|r_1r_2|, 2|r_2r_0|$ and $2|r_0r_1|$, respectively, acting on F as subgroups of $Mon(\mathcal{H})$. The sets of orbits

$$\mathcal{V} = F/M_0, \qquad \mathcal{E} = F/M_1, \qquad \mathcal{F} = F/M_2$$

are called the set of *hypervertices*, the set of *hyperedges* and the set of *hyperfaces* of $\mathcal{H}$, respectively, while the triple $(|r_1r_2|, |r_2r_0|, |r_0r_1|)$ is called the *type* of $\mathcal{H}$. In particular, maps have type $(k, 2, n)$ for some $k, n \in \mathbb{N}$.

A *morphism* ϕ from the hypermap $\widetilde{\mathcal{H}} = (\widetilde{F}; \widetilde{r_0}, \widetilde{r_1}, \widetilde{r_2})$ to the hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ is a function $\phi : \widetilde{F} \to F$ satisfying $\widetilde{r_i}\,\phi = \phi\, r_i$, for every $i \in \{0, 1, 2\}$. If there is a morphism from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$, then we say that $\widetilde{\mathcal{H}}$ *covers* $\mathcal{H}$ or that $\widetilde{\mathcal{H}}$ is *a cover* of $\mathcal{H}$. It follows from the definitions that any morphism is uniquely determined by the image of a flag, and that it is onto. Therefore morphisms are also called *coverings* and one-to-one coverings are called *isomorphisms*. If there is an isomorphism ϕ from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$, then ϕ^{-1} is an isomorphism from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$. In that case, the hypermaps $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ are said to be *isomorphic* and we write $\widetilde{\mathcal{H}} \cong \mathcal{H}$. An isomorphism from $\mathcal{H}$ to $\mathcal{H}$ is called an *automorphism* of $\mathcal{H}$. The set $Aut(\mathcal{H})$ of automorphisms of $\mathcal{H}$ is a (permutation) group (on the set F of flags of $\mathcal{H}$), called the *automorphism group* of $\mathcal{H}$. By definition, $Aut(\mathcal{H})$ is the centralizer of $Mon(\mathcal{H})$ in the symmetric group S_F on the set F of flags of $\mathcal{H}$. Therefore $Aut(\mathcal{H})$ acts semi-regularly on F (see, for instance, Theorem 4.2A of [4]) and hence

$$|Aut(\mathcal{H})| \leqslant |F| \leqslant |Mon(\mathcal{H})|.$$

An equality on one side implies equality on the other side. This happens if and only if the action of $Mon(\mathcal{H})$ on F is regular. In this case $\mathcal{H}$ is called a *regular hypermap*.

Any hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ gives rise to a triangulation of a closed (compact and connected) surface $\mathcal{S}$, called the *supporting surface* of $\mathcal{H}$. Triangles with vertices labelled 0, 1, and 2, correspond to the flags of $\mathcal{H}$, and the triangles x and xr_i are glued together along the respective edges with vertices labelled j and k, where $\{i, j, k\} = \{0, 1, 2\}$ (see Fig. 1).

The 1-skeleton of the triangulation is an embedded graph $\mathcal{G}$ on $\mathcal{S}$ with vertices labelled 0, 1, and 2 corresponding to hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, respectively. Hence the number of vertices of $\mathcal{G}$ is $|\mathcal{V}| + |\mathcal{E}| + |\mathcal{F}|$. As each edge is incident with two triangles and $|F|$ is the number of triangles, the number of edges of $\mathcal{G}$ is $\frac{3|F|}{2}$. From the Euler formula we conclude that the *characteristic* (of the supporting surface $\mathcal{S}$) *of* $\mathcal{H}$ is

$$\chi(\mathcal{H}) = |\mathcal{V}| + |\mathcal{E}| + |\mathcal{F}| - \frac{|F|}{2}.$$

For other representations of hypermaps see [7] or [12].

Let $x\nu_i$ be the size of the orbit of $x \in F$ by the action of the cyclic subgroup $\langle r_jr_k\rangle$ of M_i ($\{i, j, k\} = \{0, 1, 2\}$). Then $|xM_i| = 2(x\nu_i)$, and using Burnside's Lemma we get

$$\chi(\mathcal{H}) = -\frac{1}{2}\sum_{x\in F}\left(1 - \sum_{i=0}^{2}\frac{1}{x\nu_i}\right).$$

Whenever $xM_i = yM_i$, $x\nu_i = y\nu_i$, and so we can regard ν_i as a function from F/M_i to $\mathbb{N}$. The image of xM_i by ν_i is called the *valency* of (the hypervertex, the hyperedge or the hyperface) xM_i (according to $i = 0$, 1 or 2). A hypermap is called *uniform* if ν_0, ν_1 and ν_2 are constant. For a uniform hypermap $\mathcal{H}$ of type (k, m, n) we have

$$\chi(\mathcal{H}) = \frac{|F|}{2}\left(\frac{1}{k} + \frac{1}{m} + \frac{1}{n} - 1\right). \tag{1}$$

We say that $\mathcal{H} = (F; r_0, r_1, r_2)$ is *orientable* if the supporting surface $\mathcal{S}$ of $\mathcal{H}$ is orientable. Otherwise, we say that $\mathcal{H}$ is *non-orientable*. By considering the group $Mon^+(\mathcal{H}) := \langle r_1r_2, r_2r_0\rangle$, we have that $\mathcal{H}$ is orientable if and only if $Mon^+(\mathcal{H}) \neq Mon(\mathcal{H})$, in which case $Mon^+(\mathcal{H})$ is a normal subgroup of index 2 in $Mon(\mathcal{H})$ and we can give orientations to the triangles of the triangulation of $\mathcal{S}$ induced by $\mathcal{H}$ in such a way that the triangles x and xr_i have opposite orientations, $i = 0, 1, 2$.

By definition, the monodromy group $Mon(\mathcal{H})$ of a hypermap $\mathcal{H} = (F; r_0, r_1, r_2)$ is a finite quotient of the free product

$$\Delta := \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 = \langle R_0, R_1, R_2 \mid {R_0}^2, {R_1}^2, {R_2}^2\rangle.$$