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Ideals in atomic posets

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ABSTRACT

The "bottom" of a partially ordered set (poset) Q is the set Q^ℓ of its lower bounds (hence, Q^ℓ is empty or a singleton). The poset Q is said to be atomic if each element of $Q \setminus Q^\ell$ dominates an atom, that is, a minimal element of $Q \setminus Q^\ell$. Thus, all finite posets are atomic. We study general closure systems of down-sets (referred to as ideals) in posets. In particular, we investigate so-called m-ideals for arbitrary cardinals m, providing common generalizations of ideals in lattices and of cuts in posets. Various properties of posets and their atoms are described by means of ideals, polars (annihilators) and residuals, defined parallel to ring theory. We deduce diverse characterizations of atomic posets satisfying certain distributive laws, e.g. by the representation of specific ideals as intersections of prime ideals, or by maximality and minimality properties. We investigate non-dense ideals (down-sets having nontrivial polars) and semiprime ideals (down-sets all of whose residuals are ideals). Our results are constructive in that they do not require any set-theoretical choice principles.

0. Introduction

Discrete mathematics is the branch of mathematics dealing
with objects that can assume only distinct, separated values.
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This strange description raises an obvious question: what is meant by "distinct, separated values?" A precise answer is given in topology: a topological space is discrete if and only if each point has a singleton neighborhood. More generally, Alexandroff [1] called a topological (T_0) space "discrete" if every point has a least neighborhood (i.e., arbitrary intersections of open sets are open), and observed that the category of such spaces is isomorphic to the category of posets, by assigning to each poset the topology of all up- or down-sets (containing with any b all $a \geq b$ or all $a \leq b$, respectively). This is in accordance with the custom to consider major parts of order theory as belonging to discrete mathematics.

Often, a mathematical structure is regarded as having a "discrete" feature if it possesses properties very similar to those of finite structures of the same kind and admits algorithmic methods that do not rely on any non-constructive set-theoretical choice principles. For example, geometries of finite height share many properties with finite geometries. Or, rings, modules, algebras and lattices satisfying certain chain conditions admit a "discrete" treatment that fails for arbitrary structures in the respective class. Similarly, many properties of finite posets remain valid for the much larger class of well-founded posets (that is, posets in which all nonempty subsets have minimal elements), or even for atomic posets, which we are going to define in the next paragraph.

In the absence of some least upper bounds (*joins*) or greatest lower bounds (*meets*) in a poset Q, natural substitutes are *cuts* (Birkhoff [3], MacNeille [37]), that is, subsets that consist of all lower bounds of their set of upper bounds. Otherwise

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expressed, a cut is nothing but an intersection of *principal ideals* $\downarrow b = \{a \in Q \mid a \leq b\}$. The "bottom" of Q is the least cut $Q^\ell = \bigcap\{\downarrow b \mid b \in Q\}$; it is either empty or consists of the least element $\mathbf{0}$ of Q alone (if it exists). The minimal elements of $Q\backslash Q^\ell$ are the *atoms*; thus, in the absence of a least element, the atoms are merely the minimal elements of Q. Being small and disjoint, atoms may be regarded as "separated values" alluded to in the introductory quotation. In an *atomic* poset Q each element of $Q\backslash Q^\ell$ lies above an atom. All well-founded posets and, in particular, all finite posets are atomic.

In this paper, we are mainly interested in ideal-theoretic properties and various degrees of (finite or infinite) distributivity in atomic posets. Ideals are well investigated tools in ring and lattice theory, with many applications in other fields. Since the pioneering work of Scott [42] and Tarski [46] it is known that suitable choice principles together with weak forms of distributivity guarantee the existence of prime ideals or filters in lattices; one of the weakest such properties is the so-called $\mathbf{0}$-*distributivity* (Varlet [47]) or *meet-semidistributivity at* $\mathbf{0}$ (Gorbunov [23,24]), demanding a least element $\mathbf{0}$ satisfying the implication

$$a \wedge b = a \wedge c = \mathbf{0} \;\Rightarrow\; a \wedge (b \vee c) = \mathbf{0},$$

or the dual property, called $\mathbf{1}$-*distributivity* and defined by the implication

$$a \vee b = a \vee c = \mathbf{1} \;\Rightarrow\; a \vee (b \wedge c) = \mathbf{1},$$

where $\mathbf{1}$ denotes a greatest element. For posets instead of lattices, the discussion becomes more subtle, because there are various reasonable notions of ideals, of distributivity, of primeness, etc., and the absence of joins or meets may cause serious problems. This area of order theory was investigated in a series of papers by Erné [10–20] and independently by Chajda, Halaš, Larmerová, Rachůnek, Niederle [5,6,26,27,29,30,36,39], and later by Joshi, Kharat, Mokbel, Mundlik, Waphare [31,32,34,49,50] and many others.

Specifically, we investigate so-called *m-ideals*, including many ideal concepts considered in order and lattice theory: if m is a cardinal greater than 1, then an m-ideal of a poset is a subset I that contains with any *m-small subset* B (any subset of cardinality smaller than m) of I the cut generated by B (the intersection of all principal ideals containing B). For the least infinite cardinal ω, the ω-ideals are the ideals in the sense of Frink [21]. If the entire poset is m-small then the m-ideals are just the cuts. The m-ideals of a poset form a closure system, hence a complete lattice (ordered by inclusion), referred to as the *m-ideal completion* (Rosický [41], Erné [10,18,19]). The m-ideal completions are special instances of so-called *standard completions* of a poset, i.e. closure systems consisting of certain down-sets and containing at least all principal ideals [13,20]. The standard completions of a poset Q together with the embeddings $x \mapsto \downarrow x$ form a system of representatives for all join-dense embeddings of Q in complete lattices L (i.e., each element of L is a join of image elements), justifying the nomenclature from a categorical point of view (Banaschewski [2], Erné [13]). Standard completions with least element Q^ℓ are referred to as *ideal completions*, and their members as *ideals* in the widest sense. Thus, like the word "closed" in topology, the word "ideal" refers to certain specified closure systems in order theory. We shall extend great parts of the theory of m-ideal completions to arbitrary ideal completions.

The prime ideal theorems for Boolean algebras and distributive lattices (Stone [44,45]), for arbitrary lattices (Rav [40]), and for general algebras and posets (Erné [14,16,17]) all require choice principles equivalent to the Ultrafilter Principle, which postulates that every proper set-theoretical filter be contained in an ultrafilter—a principle not provable in choice-free set theory. As a "constructive" substitute, the presence of enough atoms or coatoms (dual atoms) allows to work in an entirely choice-free setting, because then weak distributivity assumptions give rise to certain prime filters or ideals, respectively. We shall study that phenomenon in arbitrary posets. Thus, we include not only a study of general concepts of distributivity but also of prime elements and ideals in posets, generalizing the familiar case of lattices: a proper ideal P of a poset Q is *prime* if $\downarrow a \cap \downarrow b \subseteq P$ implies $a \in P$ or $b \in P$, and *completely prime* if $Q\backslash P$ has a least element; and an element p of Q is *prime* or *completely prime*, respectively, if so is the principal ideal $\downarrow p$.

An element a of Q is $\perp$-*m-distributive* if its *polar* $a^\perp = \{b \in Q \mid \downarrow a \cap \downarrow b = Q^\ell\}$ is an m-ideal; in complete lattices with $\mathbf{0} = \bigvee \emptyset$, this condition amounts to $a \wedge \bigvee B = \mathbf{0}$ for any m-small subset B of Q with $a \wedge b = \mathbf{0}$ for all $b \in B$, hence to a generalization of $\mathbf{0}$-distributivity. But $\perp$-m-distributivity is also closely related to pseudocomplementation, because the *pseudocomplement* of an element is the greatest element of its polar (provided such an element exists). Indeed, a poset (i.e. each of its elements) is $\perp$-m-distributive if and only if its m-ideal completion is pseudocomplemented or, equivalently, $\perp$-m-distributive [19]. We add further necessary and sufficient conditions for an ideal completion $\mathcal{C}$ to be atomic and pseudocomplemented, for example, that the *polars* $A^\perp = \{b \in Q \mid A \cap \downarrow b \subseteq Q^\ell\}$ of all subsets A have irredundant representations as a meet of completely prime ideals; or, that $\mathcal{C}$ contains all polars and the polarization map, which sends each subset to its polar, induces a dual isomorphism between the power set of the atoms and the *skeleton* $\{D^\perp \mid D \in \mathcal{C}\}$ of $\mathcal{C}$. Furthermore, in that situation, we find that the maximal non-dense ideals (where an ideal is *non-dense* if its polar is not the least one) are not only the prime non-dense ideals, but also the polars of atoms and also the minimal completely prime ideals.

In the last section, we study *semiprime ideals*, i.e. members D of ideal completions $\mathcal{C}$ such that all *residuals* $D : a = \{b \in Q \mid \downarrow a \cap \downarrow b \subseteq D\}$ belong to $\mathcal{C}$. We shall see that an ideal has an irredundant representation as an intersection of completely prime ideals if and only if it is semiprime and its set-theoretical complement is atomic or has a least element. As a consequence, a well-founded poset is *m-distributive*, i.e., each residual is an m-ideal, if and only if each m-ideal is an intersection of prime (and even of completely prime) m-ideals.