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On a local similarity of graphs

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ABSTRACT

We say that two graphs G and H, having the same number of vertices n, are k-similar if they contain a common induced subgraph of order k. We will consider the following question: how large does n need to be to ensure at least one k-similar pair in any family of l graphs on n vertices? We will present various lower and upper bounds on n. In particular, we will prove that for $l = 3$, n equals the Ramsey number $R(k, k)$. Last but not least we will determine the exact values of n for $k = 3$, $k = 4$ and all l.

1. Introduction

In this paper all graphs are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph and $\overline{G}$ the complement of G. We assume that the reader is familiar with standard graph-theoretic terminology and refer the readers to standard texts from graph theory for any notation that is not defined here.

We say that two graphs G and H, having the same number of vertices n, are k-similar if they contain a common induced subgraph of order k. Assume that $l \geq 3$.

Definition 1. Let $\eta(k, l)$ be the smallest n such that in any family of l graphs on n vertices there exists a k-similar pair of graphs.

The problem of setting the value of $\eta(k, l)$ is naturally linked to the question of how much l graphs may be different from each other.

In this article we are considering the problem of finding the value $\eta(k, l)$. To the best of our knowledge no problem of this sort has been studied before. However somewhat similar questions was put by Chung, Erdös and Spencer in [4] and by Chung, Erdös, Graham, Ulam and Yao in [3]. The authors of those articles were interested in finding a common induced subgraph of two dense graphs. For two graphs G and H they studied the properties of the function $U(G, H)$ which is the least integer t such that $E(G)$ can be partitioned into $E_1, \ldots, E_t$, and $E(H)$ can be partitioned into $E'_1, \ldots, E'_t$ in such a way that the graphs formed by E_i and E'_i are isomorphic for each i. Some new considerations were presented by other authors, including Bollobás, Kittipassorn, Narayanan and Scott [2] and Lee, Loh and Sudakov [6]. While these are not directly related to the problem at hand, they are similar in nature, and provide further justification for studying the function $\eta(k, l)$.

An additional motivation for studying $\eta(k, l)$ is the fact that it is closely related to the Ramsey number. The Ramsey number $R(k, k)$ is the minimum number n such that any graph G on n vertices contains either a k-vertex clique K_k, or an independent set of size k denoted by $\overline{K_k}$ (see [7] for known values, properties and references to these numbers). It will be shown that $\eta(k, 3) = R(k, k)$, therefore the number $\eta(k, l)$ might be considered a non-trivial generalization of the Ramsey number.

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Table 1
Values of $\eta(k, l)$ for $k = 3$ and $k = 4$.

l	3	4	5	6	7	8	9	10	11	≥12
$k = 3$	6	5	3	3	3	3	3	3	3	3
$k = 4$	18	10	7	6	6	5	5	5	5	4

In the next section we will present the connections of $\eta(k, l)$ to the Ramsey number. Then, in the following section, we will show various lower bounds on $\eta(k, l)$. Section 4 is devoted to the case of large $l = l(k)$. In the last section we will give exact results for small values of k which are summarized in Table 1.

2. Relation to the Ramsey number

Theorem 2. *Let $k \geq 3$. Then*

$$\eta(k, 3) = R(k, k).$$

Since $\eta(k, 3) = R(k, k)$ and $\eta(k, l) \geq \eta(k, l')$ for $l < l'$, then we immediately obtain an important consequence which is an important consequence of Theorem 2.

Corollary 3. *Let $k, l \geq 3$. Then the number $\eta(k, l)$ is a well-defined finite number.*

The Ramsey number gives also an upper bound for $\eta(k, l)$ if $l \geq 2k + 1$. It is depicted by the following theorem.

Theorem 4. *Let $k \geq 3$. Then $\eta(k, 2k + 1) \leq R(k - 1, k - 1)$.*

Now we will prove both theorems.

Proof of Theorem 2. Denote by R_k the Ramsey graph, i.e. a graph with maximum possible number of vertices n, no clique of size k, and no independent set of size k. By the definition $|V(R_k)| = R(k, k) - 1$.

For the lower bound $\eta(k, 3) \geq R(k, k)$, assume that $n = R(k, k) - 1$ and consider the graphs $G_1 = K_n$, $G_2 = \overline{K_n}$ and $G_3 = R_k$. Observe that each possible k-vertex subgraph of G_1 and G_2 is K_k and $\overline{K_k}$, respectively. Moreover, $G_3 = R_k$ contains neither K_k nor $\overline{K_k}$. Therefore among G_1, G_2, G_3 there is no k-similar pair of graphs.

For the upper bound $\eta(k, 3) \leq R(k, k)$, let us consider three arbitrary graphs G_1, G_2, G_3 such that $|V(G_1)| = |V(G_2)| = |V(G_3)| = n$ and $n \geq R(k, k)$. By the definition of the Ramsey number, we have that each G_i contains K_k or $\overline{K_k}$. Therefore by the Pigeonhole Principle, among G_1, G_2, G_3 there are two graphs which contain K_k or two graphs which contain $\overline{K_k}$. Those graphs form a k-similar pair of graphs. □

Proof of Theorem 4. Consider any $2k+1$ graphs $G_1, G_2, \ldots, G_{2k+1}$ of order $R(k-1, k-1)$. Since $|V(G_i)| = R(k-1, k-1)$, then each of graphs $G_1, G_2, \ldots, G_{2k+1}$ contains either a clique or an independent set of order $k - 1$. By the Pigeonhole Principle, at least $k + 1$ among them contain K_{k-1} or at least $k + 1$ of them contain $\overline{K_{k-1}}$. Without loss of generality assume that $G_1, G_2, \ldots, G_{k+1}$ have K_{k-1} as a subgraph. For each $1 \leq i \leq k + 1$ fix a vertex $v_i \in V(G_i) - K_{k-1}$. In each G_i for $1 \leq i \leq k + 1$ consider the subgraph induced by the vertices of the K_{k-1} and v_i. Since v_i may be joined by $0, 1, \ldots,$ or $k - 1$ edges of K_{k-1}, it follows from the Pigeonhole Principle that there are two graphs with v_i having the same degree to the clique, thus giving a k-similar pair of graphs among $G_1, G_2, \ldots, G_{k+1}$. □

3. Lower bounds

We present here some lower bounds for different range of parameters.

Theorem 5. *Let $k \geq 3$. Then $\eta(k, 4) > (k - 1)^2$.*

This bound is sometimes tight, as evidenced in Section 5 (see $\eta(3, 4) = 5$ or $\eta(4, 4) = 10$).

Theorem 6. *Let $k, l \geq 3$ and $t \geq 1$. Then*

$$\eta(tk, l) > t\eta(\lceil k/t \rceil, tl) - t.$$

The above theorems are constructive. The following one relies on the probabilistic method.

Theorem 7. *Let $k, l \geq 3$ then*

$$\eta(k, l) \geq \frac{(k-2)^{(k-2)/(2k-2)} 2^{k/4}}{e^{1/2} k^{1/(k-1)} l^{1/(2k-2)}}.$$

Remark 8. Using the first moment method one may obtain the following, similar result

$$\eta(k, l) > \frac{k^{1/2} 2^{(k-1)/4}}{e^{1/2} l^{1/k}}, \quad \text{for } k, l \geq 3.$$