Discrete Mathematics 338 (2015) 994–999

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Ramsey numbers of trees versus fans

Yanbo Zhang [a,b], Hajo Broersma [b], Yaojun Chen [a,*]

[a] Department of Mathematics, Nanjing University, Nanjing 210093, PR China

[b] Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

ARTICLE INFO

Article history:
Received 17 July 2014
Received in revised form 23 January 2015
Accepted 24 January 2015
Available online 14 February 2015

Keywords:
Ramsey number
Tree
Star
Fan

ABSTRACT

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or the complement of G contains G_2 as a subgraph. Let T_n be a tree of order n, S_n a star of order n, and F_m a fan of order $2m + 1$, i.e., m triangles sharing exactly one vertex. In this paper, we prove that $R(T_n, F_m) = 2n - 1$ for $n \geq 3m^2 - 2m - 1$, and if $T_n = S_n$, then the range can be replaced by $n \geq \max\{m(m-1) + 1, 6(m-1)\}$, which is tight in some sense.

1. Introduction

In this paper we deal with finite simple graphs only. For a nonempty proper subset $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph induced by S and $V(G) - S$, respectively. Let $N_S(v)$ be the set of all the neighbors of a vertex v that are contained in S, $N_S[v] = N_S(v) \cup \{v\}$ and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For two vertex-disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes their disjoint union and $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. We use mG to denote the union of m vertex-disjoint copies of G. A path, a star, a tree, a cycle and a complete graph of order n are denoted by P_n, $S_n = K_1 + (n-1)K_1$, T_n, C_n and K_n, respectively. A book $B_n = K_2 + nK_1$, i.e., it consists of n triangles sharing exactly one common edge, and a fan $F_n = K_1 + nK_2$, i.e., it consists of n triangles sharing exactly one common vertex. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of a graph G.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. If both G_1 and G_2 are complete graphs, then $R(G_1, G_2)$ is called a classical Ramsey number, otherwise it is called a generalized Ramsey number. Because of the extreme difficulty encountered in the determination of classical Ramsey numbers, Chvátal and Harary [10–12] in a series of papers suggested studying generalized Ramsey numbers, both for their own sake, and for the light they might shed on classical Ramsey numbers. The following is a celebrated early result on generalized Ramsey numbers due to Chvátal.

Theorem 1 (*Chvátal [9]*). *$R(T_n, K_m) = (n-1)(m-1) + 1$ for all positive integers m and n.*

Let H be a connected graph of order p, $\chi(G)$ the chromatic number of G and $s(G)$ the chromatic surplus of G, i.e., the minimum number of vertices in some color class under all proper vertex colorings with $\chi(G)$ colors. Based on Chvátal's result, Burr [4]

* Corresponding author.
E-mail address: yaojunc@nju.edu.cn (Y. Chen).

http://dx.doi.org/10.1016/j.disc.2015.01.030

established the following general lower bound for $R(H, G)$ when $p \geq s(G)$: $R(H, G) \geq (p-1)(\chi(G)-1)+s(G)$. He also defined H to be G-good in case equality holds in this inequality. By Theorem 1, it is easy to see that T_n is K_m-good. This raises the natural questions whether and when T_n is G-good if G consists of ℓ complete graphs K_m sharing exactly one vertex. A special case of the question is whether T_n is F_ℓ-good. Another natural question is for what graphs G, T_n is G-good.

In 1982, Burr et al. determined the Ramsey numbers of sufficiently large trees versus odd cycles, by showing that T_n is C_m-good for odd $m \geq 3$ and $n \geq 756m^{10}$.

Theorem 2 (*Burr et al. [5]*). *$R(T_n, C_m) = 2n-1$ for odd $m \geq 3$ and $n \geq 756m^{10}$.*

In 1988, Erdős et al. confirmed the Ramsey numbers of relatively large trees versus books, by showing that T_n is B_m-good for $n \geq 3m-3$, a result that we will use in our proof of Lemma 2 in the next section.

Theorem 3 (*Erdős et al. [13]*). *$R(T_n, B_m) = 2n-1$ for $n \geq 3m-3$.*

Other results on Ramsey numbers concerning trees can be found in [1–3,6–8,14], see [15] for a survey. In this paper, we first show that S_n is F_m-good for all integers $n \geq \max\{m(m-1)+1, 6(m-1)\}$, by proving the following result.

Theorem 4. *$R(S_n, F_m) = 2n-1$ for $n \geq m(m-1)+1$ and $m \neq 3, 4, 5$, and the lower bound $n \geq m(m-1)+1$ is best possible. $R(S_n, F_m) = 2n-1$ for $n \geq 6(m-1)$ and $m = 3, 4, 5$.*

We postpone the proof of Theorem 4 to the last section. Next we show that T_n is F_m-good for positive integers $n \geq 3m^2-2m-1$, which is the main theorem of our paper.

Theorem 5. *$R(T_n, F_m) = 2n-1$ for all integers $n \geq 3m^2-2m-1$.*

We also postpone the proof of Theorem 5 to the last section. We next show that the following more general result can be obtained from Theorem 5 by induction.

Corollary 1. *$R(T_n, K_{\ell-1}+mK_2) = \ell(n-1)+1$ for $\ell \geq 2$ and $n \geq 3m^2-2m-1$.*

Proof. By Theorem 5, the statement is valid for $\ell = 2$. Assume that $k \geq 3$ and that the statement holds for all integers ℓ with $2 \leq \ell < k$. We prove that it also holds for $\ell = k$.

Since kK_{n-1} contains no T_n and its complement contains no K_{k+1}, hence no $K_{k-1}+mK_2$, we have $R(T_n, K_{k-1}+mK_2) \geq k(n-1)+1$. Let G be a graph of order $k(n-1)+1$. If $\delta(G) \geq n-1$, then by the following folklore lemma that is straightforward to prove using a Greedy approach, G contains T_n and the proof is complete. We present the lemma in a more specific form since we will use it in this form in the sequel.

Lemma 1. *Let G be a graph with $\delta(G) \geq k$, and let $u \in V(G)$. Let T be a tree of order $k+1$ with $v \in V(T)$. Then T can be embedded into G in such a way that v is mapped to u.*

Let us now assume that $\delta(G) \leq n-2$. Then $\Delta(\overline{G}) \geq (k-1)(n-1)+1$. Let v be a vertex with $d_{\overline{G}}(v) = \Delta(\overline{G})$. Then, by the induction hypothesis either $G[N_{\overline{G}}(v)]$ contains a T_n, or $\overline{G}[N_{\overline{G}}(v)]$ contains a $K_{k-2}+mK_2$, which together with v forms a $K_{k-1}+mK_2$ in $\overline{G}$. This completes the proof of Corollary 1. ■

We finish this section by posing a conjecture on the best possible lower bound for n for which T_n is F_m-good.

Conjecture 1. *$R(T_n, F_m) = 2n-1$ for $n \geq m^2-m+1$.*

Let G be any given graph. It is believed that $R(T_n, G) \leq R(S_n, G)$ in general, and all known results point in this direction. Based on this and Theorem 4, we believe that the above conjecture holds, at least for $m \geq 6$.

2. Two preliminary lemmas

In the next section we use the following lemma in our proof of Theorem 4. It is the special case of the statement of Theorem 4 when $m = 2$.

Lemma 2. *$R(S_n, F_2) = 2n-1$ for $n \geq 3$.*

Proof. The lower bound $R(S_n, F_2) \geq 2n-1$ is implied by the fact that $2K_{n-1}$ contains no S_n and its complement contains no triangle, hence no F_2. It remains to prove that $R(S_n, F_2) \leq 2n-1$ for $n \geq 3$.

Let G be a graph of order $2n-1$. Suppose that G contains no F_2 and $\overline{G}$ has no S_n. Then $\Delta(\overline{G}) \leq n-2$ and so $\delta(G) \geq n$. By Theorem 3, G contains B_2. Let $x_1x_2x_3x_4$ be a C_4 with diagonal x_2x_4 in G. Set $X = \{x_1, x_2, x_3, x_4\}$ and $Y = V(G)-X$. If $n = 3$, then $|Y| = 1$ and the vertex in Y has at least three neighbors in X, and so G has F_2, a contradiction. Hence, $n \geq 4$. If $x_1x_3 \in E(G)$, then $N_Y(x_i) \cap N_Y(x_j) = \emptyset$ for $1 \leq i < j \leq 4$, otherwise G contains F_2. Thus, we have $4(n-2) \leq \sum_{k=1}^{4} d_Y(x_k)+4 \leq 2n-1$, which implies that $n \leq 3$, a contradiction. If $x_1x_3 \notin E(G)$, then since G has no F_2, we get that $N_Y(x_1) \cap N_Y(x_i) = \emptyset$ for $i = 2, 4$ and $N_Y(x_1)$ is an independent set of cardinality at least $n-2$. In this case, we have $d(y) \leq n-1$ for any $y \in N_Y(x_1)$, which contradicts that $\delta(G) \geq n$. ■