Discrete Mathematics 338 (2015) 1025–1028

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On colorings of variable words

Konstantinos Tyros

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

ARTICLE INFO

Article history:
Received 10 September 2014
Received in revised form 23 January 2015
Accepted 24 January 2015
Available online 17 February 2015

Keywords:
Ramsey Theory
Graham–Rothschild theorem
Hales–Jewett Theorem
Variable words

ABSTRACT

In this note, we prove that the base case of the Graham–Rothschild Theorem, i.e., the one that considers colorings of the (1-dimensional) variable words, admits bounds in the class $\mathcal{E}^5$ of Grzegorczyk's hierarchy.

1. Introduction

The Graham–Rothschild Theorem [4] is a generalization of the well known Hales–Jewett Theorem [5] that considers colorings of m-parameter sets instead of constant words. The best known bounds for the Graham–Rothschild Theorem are due to S. Shelah [7] and belong to the class $\mathcal{E}^6$ of Grzegorczyk's hierarchy. In this note we consider the "base" case of the Graham–Rothschild Theorem, that concerns colorings of (1-dimensional) variable words. We obtain bounds for this base case in $\mathcal{E}^5$ of Grzegorczyk's hierarchy. Shelah's argument makes an iterated use of the Hales–Jewett Theorem, which has bounds in $\mathcal{E}^5$ (see [7]), yielding even for the base case of the Graham–Rothschild Theorem bounds in $\mathcal{E}^6$. Roughly speaking, the improvement of the bounds is obtained by iterating a modified version of Shelah's insensitivity lemma instead of the Hales–Jewett Theorem. Although the proof is an appropriate modification of S. Shelah's proof for the Hales–Jewett Theorem, it is streamlined and independent.

The base case of the Graham–Rothschild Theorem is of particular interest, since it is the one needed for the proof of the density Hales–Jewett Theorem in [2]. Moreover, it has as an immediate consequence the finite version of the Carlson–Simpson Theorem on the left variable words and therefore the finite version of the Halpern–Läuchli theorem for level products of homogeneous trees (see also [8]).

To state the result of this note, we need some pieces of notation. Let k and n be positive integers. By $[k]$ we denote the set $\{1, \dots, k\}$ and $[k]^n$ the set of all sequences $(a_0, \dots, a_{n-1})$ of length n taking values in $[k]$. We view $[k]$ as a finite alphabet and the elements of $[k]^n$ as words. Thus, by the term *word over k of length n* we mean an element of $[k]^n$. Also let m be a positive integer and $v, v_0, \dots, v_{m-1}$ distinct symbols not belonging to $[k]$. We view these symbols as variables. A variable word $w(v)$ over k is a sequence in $[k] \cup \{v\}$, where the variable v occurs at least once. More generally, an m-dimensional variable word $w(v_0, \dots, v_{m-1})$ over k is a sequence in $[k] \cup \{v_0, \dots, v_{m-1}\}$ such that each v_j occurs at least once and they are in block position, meaning that if $w(v_0, \dots, v_{m-1})$ is of the form $(x_0, \dots, x_{n-1})$ then $\max\{i : x_i = v_j\} < \min\{i : x_i = v_{j+1}\}$ for all $0 \leqslant j < m - 1$. Clearly, every variable word can be viewed as a 1-dimensional variable word.

Let k, m be positive integers and $w(v_0, \dots, v_{m-1})$ an m-dimensional variable word over k. For every sequence of symbols $\mathbf{x} = (x_i)_{i=0}^{m-1}$ of length m we denote by $w(\mathbf{x})$ the sequence resulting by substituting each occurrence of v_i by x_i for all

E-mail address: k.tyros@warwick.ac.uk.

http://dx.doi.org/10.1016/j.disc.2015.01.031

$0 \leqslant i < m$. Observe that $w(\mathbf{x})$ is an m'-dimensional variable word, for some $m' \leqslant m$, if and only if $\mathbf{x}$ is an m'-dimensional variable word. In particular, $w(\mathbf{x})$ is a variable word if and only if $\mathbf{x}$ is a variable word. An m'-dimensional variable word is called reduced by $w(v_0, \ldots, v_{m-1})$ if it is of the form $w(\mathbf{x})$ for some m'-dimensional variable word $\mathbf{x}$ of length m.

Theorem 1. *For every triple of positive integers k, m, r there exists a positive integer n_0 with the following property. For every integer n with $n \geqslant n_0$ and every r-coloring of all the variable words over k of length n, there exists an m-dimensional variable word $w(v_0, \ldots, v_{m-1})$ over k of length n such that the set of all variable words over k reduced by $w(v_0, \ldots, v_{m-1})$ is monochromatic. We denote the least such n_0 by $GR(k, m, r)$.*

Moreover, the numbers $GR(k, m, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^5$ of Grzegorczyk's hierarchy.

2. The Hindman Theorem

The case "$k = 1$" of Theorem 1 follows by the finite version of Hindman's theorem [6]. To state it we need some pieces of notation.

Let n, m, d be positive integers with $d \leqslant m \leqslant n$. We denote by $\mathcal{F}(n)$ the set of all non-empty subsets of $\{0, \ldots, n-1\}$. A finite sequence $\mathbf{s} = (s_i)_{i=0}^{m-1}$ in $\mathcal{F}(n)$ is called block if $\max s_i < \min s_{i+1}$ for all $0 \leqslant i < m-1$. We denote the set of all block sequences of length m in $\mathcal{F}(n)$ by $\mathrm{Block}^m(n)$. For every $\mathbf{s} = (s_i)_{i=0}^{m-1}$ in $\mathrm{Block}^m(n)$ we define the set of nonempty unions of $\mathbf{s}$ to be

$$\mathrm{NU}(\mathbf{s}) = \Big\{ \bigcup_{i \in t} s_i : t \text{ is a nonempty subset of } \{0, \ldots, m-1\} \Big\}.$$

We say that a block sequence $\mathbf{t} = (t_i)_{i=0}^{d-1}$ in $\mathcal{F}(n)$ is a block subsequence of $\mathbf{s}$ if $t_i \in \mathrm{NU}(\mathbf{s})$ for all $0 \leqslant i < d$. The finite version of Hindman's theorem is stated as follows.

Theorem 2. *For every pair m, r of positive integers, there exists a positive integer n_0 with the following property. For every finite block sequence $\mathbf{s}$ of nonempty finite subsets of $\mathbb{N}$ of length at least n_0 and every coloring of the set $\mathrm{NU}(\mathbf{s})$ with r colors, there exists a block subsequence $\mathbf{t}$ of $\mathbf{s}$ of length m such that the set $\mathrm{NU}(\mathbf{t})$ is monochromatic. We denote the least n_0 satisfying the above property by $\mathrm{H}(m, r)$.*

Moreover, the numbers $\mathrm{H}(m, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^4$ of Grzegorczyk's hierarchy.

This finite version follows by the disjoint union theorem [4,9] and Ramsey's theorem. The bounds for the disjoint union theorem given in [9], as well as, the bound for the Ramsey numbers given in [3] are in $\mathcal{E}^4$. Using these bounds, one can see that the numbers $\mathrm{H}(m, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^4$ of Grzegorczyk's hierarchy. We refer the interested reader to [1] for further details.

3. Insensitivity

The proof of Theorem 1 proceeds by induction on k. The main notion that helps us to carry out the inductive step of the proof is an appropriate modification of Shelah's insensitivity (see Definition 3 below).

First, let us introduce some additional notation. Let k, m, n be positive integers with $m \leqslant n$ and $w(v_0, \ldots, v_{m-1})$ be an m-dimensional variable word over k of length n. We denote by $W_v^k(n)$ the set of all variable words over k of length n, while by $W_v^k\big(w(v_0, \ldots, v_{m-1})\big)$ the set of all variable words over k reduced by $w(v_0, \ldots, v_{m-1})$. If $w = w(v_0, \ldots, v_{m-1}) = (x_i)_{i=0}^{n-1}$, for every $j = 0, \ldots, m-1$ we set

$$\mathrm{supp}_w(v_j) = \{i \in \{0, \ldots, n-1\} : x_i = v_j\}.$$

We consider the following analogue of Shelah's insensitivity.

Definition 3. Let k, m, n be positive integers with $m \leqslant n$. Also let $w(v_0, \ldots, v_{m-1})$ be an m-dimensional variable word over $k+1$ of length n and a, b in $[k+1]$ with $a \neq b$.

(i) We say that two variable words $\mathbf{x} = (x_i)_{i=0}^{n-1}$ and $\mathbf{y} = (y_i)_{i=0}^{n-1}$ over $k+1$ of length n are (a, b)-equivalent if for every e in $[k+1] \cup \{v\} \setminus \{a, b\}$, we have that $x_i = e$ if and only if $y_i = e$ for all i in $\{0, \ldots, n-1\}$.

(ii) We say that a coloring c of $W_v^{k+1}(n)$ is (a, b)-insensitive over $w(v_0, \ldots, v_{m-1})$ if for every pair $\mathbf{x}, \mathbf{y}$ of (a, b)-equivalent words over $k+1$ of length m, we have that $c\big(w(\mathbf{x})\big) = c\big(w(\mathbf{y})\big)$.

We prove the following analogue of Shelah's insensitivity lemma.

Lemma 4. *For every triple k, m, r of positive integers there exists a positive integer n_0 satisfying the following. For every integer n with $n \geqslant n_0$, every a, b in $[k+1]$ with $a \neq b$ and every r-coloring c of $W_v^{k+1}(n)$ there exists an m-dimensional variable word $w(v_0, \ldots, v_{m-1})$ over $k+1$ of length n such that c is (a, b)-insensitive over $w(v_0, \ldots, v_{m-1})$. We denote the least such n_0 by $\mathrm{Sh}_v(k, m, r)$.*

Finally, the numbers $\mathrm{Sh}_v(k, m, r)$ are upper bounded by a primitive recursive function belonging to the class $\mathcal{E}^4$ of Grzegorczyk's hierarchy.