Discrete Mathematics 338 (2015) 1029–1041

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Dichotomies properties on computational complexity of S-packing coloring problems

Nicolas Gastineau *

LE2I, UMR CNRS 6306, Université de Bourgogne, France
Université de Lyon, CNRS, Université Lyon 1, LIRIS, UMR5205, F-69622, France

ARTICLE INFO

Article history:
Received 17 July 2014
Received in revised form 22 January 2015
Accepted 23 January 2015
Available online 16 February 2015

Keywords:
NP-complete problem
Graph
Packing chromatic number
S-packing chromatic number
d-distance coloring

ABSTRACT

This work establishes the complexity class of several instances of the S-packing coloring problem: for a graph G, a positive integer k and a nondecreasing list of integers $S = (s_1, \ldots, s_k)$, G is *S-colorable* if its vertices can be partitioned into sets S_i, $i = 1, \ldots, k$, where each S_i is an s_i-packing (a set of vertices at pairwise distance greater than s_i). In particular we prove a dichotomy between NP-complete problems and polynomial-time solvable problems for lists of at most four integers.

1. Introduction

We consider only finite undirected connected graphs. For a graph G, an *i-packing* is a set $X_i \subseteq V(G)$ such that for any distinct pair $u, v \in X_i$, $d_G(u, v) > i$, where $d_G(u, v)$ denotes the distance between u and v. We will use X_i to refer to an i-packing in a graph G. For a nondecreasing sequence of positive integers $S = \{s_i | i > 0\}$, a *S–k-(packing)-coloring* of G is a partition of $V(G)$ into sets $S_1, \ldots, S_k$, where each S_i is an s_i-packing. For a nondecreasing list of k integers $S' = (s_1, \ldots, s_k)$, an *S'-(packing)-coloring* of G is an S–k-coloring of G for a sequence S which begins with a list S'. A graph G is *S-colorable* if there exists an S-coloring of G.

The *S-COL* decision problem consists in determining, for fixed S, if G is S-colorable, for a graph G as input.

By $|S|$, we denote the size of a list S, and by s_i we denote the ith element of a list. Let $S = (s_1, s_2, \ldots)$ and $S' = (s'_1, s'_2, \ldots)$ be two nondecreasing lists of integers with $|S| = |S'|$. We define an order on the lists by $S \le S'$ if $s_i \ge s'_i$, for every integer i, $1 \le i \le |S|$. Note that if $S \le S'$, G is S-colorable implies G is S'-colorable.

In this article, for a $(s_1, s_2, \ldots)$-coloring of a graph, we prefer to map vertices to the color multi-set $\{s_1, s_2, \ldots\}$ even if two colors can be denoted by the same number. This notation allows the reader to directly see to which type of packing the vertex belongs depending on its color. When needed, we will denote colors of vertices in different i-packings by $i_a, i_b, \ldots$.

Let S_d^k be a list only containing k integers d. The problem S_1^k-COL corresponds to the k-coloring problem which is known to be NP-complete for $k \ge 3$. The S-coloring generalizes coloring with distance constraints like the *packing coloring* or the *distance coloring* of a graph. We denote by P-COL, the problem $(1, 2, \ldots, k)$-COL for a graph G and an integer k (with G and k as input). The packing chromatic number [9] of G is the least integer k such that G is $(1, 2, \ldots, k)$-colorable. A series of works [4,6,8,9] considered the packing chromatic number of infinite grids. The d-distance chromatic number [13] of G is the

* Correspondence to: LE2I, UMR CNRS 6306, Université de Bourgogne, France.
E-mail address: Nicolas.Gastineau@u-bourgogne.fr.

http://dx.doi.org/10.1016/j.disc.2015.01.028
0012-365X/

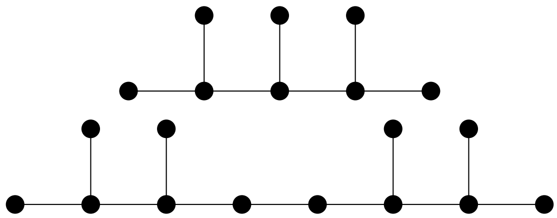

Fig. 1. The trees T_0 (on the top) and T_1 (on the bottom).

least integer k such that G is S_d^k-colorable. Initially, the concept of S-coloring has been introduced by Goddard et al. [9] and Fiala et al. [7]. The S-coloring problem was considered in other papers [10,11].

The S-coloring problem, with $|S| = 3$ has been introduced by Goddard et al. [9] in order to determine the complexity of the packing chromatic number when $k = 4$. Moreover, Goddard and Xu [10] have proven that for $|S| = 3$, S-COL is NP-complete if $s_1 = s_2 = 1$ or if $s_1 = 1$ and $s_2 = s_3 = 2$ and polynomial-time solvable otherwise. About the complexity of S-COL, Fiala et al. [7] have proven that P-COL is NP-complete for trees and Argiroffo et al. [1,2] have proven that P-COL is polynomial-time solvable on some classes of graphs.

In the second section, for a list S of three integers, we determine the family of S-colorable trees. Moreover, we determine dichotomies on cubic graphs, subcubic graphs and bipartite graphs. In the third section, we determine polynomial-time solvable and NP-complete instances of S-COL, for unfixed size of lists. We use these results to determine a dichotomy between NP-complete instances and polynomial-time solvable instances of S-COL for $|S| \leq 4$.

Note that for any nondecreasing list of integers S, we have S-COL in NP.

2. Complexity of S-COL for a list of three integers and several classes of graphs

This section is dedicated to the proofs of two theorems: Theorems 2.1 and 2.3. In [10], the family of S-colorable graphs, for $|S| = 3$, is described in the case S-COL is polynomial-time solvable. Using the properties of these families of graphs [10], it is easy to determine the S-colorable trees for $|S| = 3$ and $S \neq (1, 2, 2)$. In Theorem 2.1, we determine the $(1, 2, 2)$-colorable trees by giving a characterization by forbidding subtrees. The following definition gives a construction of this family of forbidden subtrees.

Definition 2.1. Let T_0 and T_1 be the trees from Fig. 1. We define a family of trees $\mathcal{T}$ as follows:

(i) $T_0 \in \mathcal{T}$, $T_1 \in \mathcal{T}$;
(ii) if $T \in \mathcal{T}$, then the tree T', obtained from T by removing an edge between two vertices u and v of degree 2, by adding four vertices u_1, u_2, u_3 and w and by adding the edges uu_1, u_1u_2, u_2u_3, u_3v and u_2w, is in $\mathcal{T}$.

We can note that the tree T_0 does not contain two adjacent vertices of degree 2. We will prove:

Theorem 2.1. *A tree T is $(1, 2, 2)$-colorable if and only if it does not contain a tree from $\mathcal{T}$.*

The following theorem by Goddard and Xu [10] establishes a dichotomy between NP-complete problems and polynomial-time solvable problems for $|S| = 3$.

Theorem 2.2 ([10]). *Let k be a positive integer. The problems $(1, 1, k)$-COL and $(1, 2, 2)$-COL are both NP-complete. Except these problems, S-COL is polynomial-time solvable for $|S| = 3$.*

For $(1, 1, k)$-COL, the authors of [10] provided a reduction from 3-COL with a graph of maximal degree Δ, the produced graph has maximal degree 2Δ. Since 3-COL is polynomial-time solvable for small Δ, the proof cannot be easily changed to have a reduction with subcubic graphs. For $(1, 2, 2)$-COL, the authors provided a reduction from NAE SAT, the produced graph has maximal degree Δ, where Δ is the maximal number of times a variable can appear positively or negatively. This reduction cannot be easily changed to have a reduction with subcubic graphs. In Theorem 2.3, we establish similar results for subcubic graphs, cubic graphs and bipartite graphs.

For different instances of S-COL, with $|S| = 3$, Table 1 summarizes the class of complexity of S-COL for different classes of graphs. We recall that every bipartite graph is $(1, 1, k)$-colorable, for k a positive integer and that every subcubic graph except K_4 is $(1, 1, 1)$-colorable by Brooks' theorem. We will prove:

Theorem 2.3. *Every instance of S-COL, for $|S| = 3$, is either polynomial-time solvable or NP-complete, for subcubic graphs, cubic graphs and bipartite graphs.*