Discrete Mathematics 338 (2015) 1042–1065

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Length lower bounds for reflecting sequences and universal traversal sequences

H.K. Dai

Computer Science Department, Oklahoma State University, Stillwater, OK 74078, USA

ARTICLE INFO

Article history:
Received 14 May 2013
Received in revised form 12 April 2014
Accepted 16 January 2015
Available online 23 February 2015

Keywords:
Graph traversal
Universal traversal sequence
Reflecting sequence

ABSTRACT

A universal traversal sequence for the family of all connected d-regular graphs of order n with an edge-labeling is a sequence of $\{0, 1, \ldots, d-1\}^*$ that traverses every graph of the family starting at every vertex of the graph. Reflecting sequences are variants of universal traversal sequences. A t-reflecting sequence for the family of all labeled chains of length n is a sequence of $\{0, 1\}^*$ that alternately visits the end-vertices and reflects at least t times in every labeled chain of the family. We present an algorithm for finding lower bounds on the lengths of reflecting sequences for labeled chains. Using the algorithm, we show a length lower bound of $19t - 214$ for t-reflecting sequences for labeled chains of length 7, which yields the length lower bounds of $\Omega(n^{1.51})$ and $\Omega(d^{0.49}n^{2.51})$ for universal traversal sequences for 2- and d-regular graphs, respectively, of n vertices, where $3 \leq d \leq \frac{n}{17} + 1$.

1. Preliminaries

Reflecting sequences are variants of universal traversal sequences, and were introduced by Tompa [16] in proving lower bounds on the lengths of universal traversal sequences. The study of universal traversal sequences is motivated by the complexity of graph traversal. Good bounds on the lengths of universal traversal sequences translate into good bounds on the time complexity of certain undirected graph traversal algorithms running in very limited space.

Aleliunas et al. [2] proved the existence of polynomial-length universal traversal sequences, yielding polynomial-time and logarithmic-space (non-uniform) deterministic algorithms for undirected graph traversal. Beame et al. [5] used variants of the jumping automaton for graphs of Cook and Rackoff [10] to study time–space tradeoffs for undirected graph traversal. Good lower bounds on the length of universal traversal sequences provide a prerequisite to proving time–space tradeoffs for traversing undirected graphs for these computational models.

1.1. Universal traversal sequences for edge-labeled undirected graphs

For a graph G, denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. For positive integers d and n with $d < n$, let $\mathcal{G}(d, n)$ be the set of all connected d-regular graphs of order n with an edge-labeling. The edge-labeling on $G \in \mathcal{G}(d, n)$ is defined as follows: Every edge $\{u, v\} \in E(G)$ is associated with two labels $l_{u,v}, l_{v,u} \in \{0, 1, \ldots, d-1\}$ such that for every $u \in V(G)$, $\{l_{u,v} \mid \{u, v\} \in E(G)\} = \{0, 1, \ldots, d-1\}$. Note that $\mathcal{G}(d, n)$ is not empty if and only if dn is even (see, for example, [15]). A sequence $U \in \{0, 1, \ldots, d-1\}^*$ can be considered as a sequence of edge-traversal commands and induces a walk starting at any vertex in every labeled graph in $\mathcal{G}(d, n)$. We say that such a sequence U traverses $G \in \mathcal{G}(d, n)$ starting at $v_0 \in V(G)$ if, by starting at v_0 and following the sequence of edge labels in U, all the vertices in G are eventually

E-mail address: dai@cs.okstate.edu.

http://dx.doi.org/10.1016/j.disc.2015.01.018
0012-365X/

Table 1
Some representatives of lower and upper bounds on $U(d, n)$.

	Bound on $U(d, n)$	Range of d	Reference
Lower bound			
	$\Omega(n^{\log_5 10})$ $(\approx\Omega(n^{1.43}))$	$d = 2$	[8]
	$\Omega(n^{\log_7 19})$ $(\approx\Omega(n^{1.51}))$	$d = 2$	This paper
	$\Omega(d^{2-\log_5 10} n^{1+\log_5 10})$ $(\approx\Omega(d^{0.57} n^{2.43}))$	$3 \le d \le n^{1-\log_{10} 5}$ $(\approx n^{0.30})$	[8]
	$\Omega(d^{2-\log_7 19} n^{1+\log_7 19})$ $(\approx\Omega(d^{0.49} n^{2.51}))$	$3 \le d \le n^{1-\log_{19} 7}$ $(\approx n^{0.34})$	This paper
	$\Omega(d^2 n^2)$	$(n^{0.34} \approx)\, n^{1-\log_{19} 7} < n \le \frac{n}{3} - 2$	[7]
	$\Omega(n^2)$	$\frac{n}{3} - 2 < d$	[3]
Upper bound			
	$O(n^3)$	$d = 2$	[1]
	$O(n^{4.03})$ (log-space constructible)	$d = 2$	[14]
	$O(dn^3 \log n)$	$3 \le d \le \frac{n}{2} - 1$	[13]
	$O(n^3 \log n)$	$\frac{n}{2} - 1 < d$	[9]

visited. The sequence U is a universal traversal sequence (UTS) for $\mathcal{G}(d, n)$ if U traverses every $G \in \mathcal{G}(d, n)$ starting at every vertex in G.

Let $U(d, n)$ denote the length of a shortest UTS for non-empty $\mathcal{G}(d, n)$, and define $U(d, n) = U(d, n + 1)$ in case $\mathcal{G}(d, n)$ is empty. The lower and upper bounds on $U(d, n)$ for various ranges of d were studied in [1–4,7,9,13,16,8,12,11], and [14]. Some representatives of these bounds on $U(d, n)$ are summarized in Table 1.

Currently, the best lower bounds on $U(d, n)$ are:

$$U(d, n) = \begin{cases} \Omega(n^{\log_7 19}) & \text{if } d = 2, \\ \Omega(d^{2-\log_7 19} n^{1+\log_7 19}) & \text{if } 3 \le d \le \dfrac{n}{17} + 1, \end{cases}$$

which were obtained using a computationally intensive approach with several optimizations in the algorithm. In this paper we present the algorithm[1] and detail the refinements and optimizations that aim to lower its time complexity.

1.2. Reflecting sequences for labeled chains

For positive integer n, a labeled chain of length n is a graph G with vertex set $V(G) = \{0, 1, \ldots, n\}$ and edge set $E(G) = \{\{i, i + 1\} \mid 0 \le i < n\}$ with an edge-labeling defined as follows: Every edge $\{i, i + 1\} \in E(G)$ is associated with two labels $l_{i,i+1}$ and $l_{i+1,i}$, each a non-empty subset of $\{0, 1\}$, such that: (1) $l_{0,1} = l_{n,n-1} = \{0, 1\}$, and (2) $l_{i,i-1}$ and $l_{i,i+1}$ form a partition of $\{0, 1\}$ for each $i \in \{1, 2, \ldots, n - 1\}$.

Denote by $\mathcal{L}(n)$ the set of all labeled chains of length n. We identify a labeled chain $G \in \mathcal{L}(n)$ with the sequence (of forward edge-labels) $\alpha = \alpha_1\alpha_2 \cdots \alpha_{n-1} \in \{0, 1\}^{n-1}$, where $l_{i,i+1} = \{\alpha_i\}$ for all $i \in \{1, 2, \ldots, n-1\}$; the sequence α is called the label of G.

Given a labeled chain $G \in \mathcal{L}(n)$, every sequence $U = U_1U_2 \cdots U_k \in \{0, 1\}^k$ where $k \ge 0$, when considered as an edge-traversal sequence starting at vertex 0 in G, determines a unique sequence $(0 = v_0, v_1, v_2, \ldots, v_k) \in \{0, 1, \ldots, n\}^{k+1}$ such that $U_i \in l_{v_{i-1},v_i}$ for all $i \in \{1, 2, \ldots, k\}$. For positive integer t, a sequence $U \in \{0, 1\}^*$ is said to reflect t times in $G \in \mathcal{L}(n)$ if the walk in G induced by U alternately visits the end-vertices n and 0 at least t times, that is, there exists a sequence of ascending indices, $0 < i_1 < i_2 < \cdots < i_t \le |U|$, such that $v_{i_{2j-1}} = n$ for all $j \in \{1, 2, \ldots, \lceil \frac{t}{2} \rceil\}$ and $v_{i_{2j}} = 0$ for all $j \in \{1, 2, \ldots, \lfloor \frac{t}{2} \rfloor\}$. The sequence U is called a t-reflecting sequence for $\mathcal{L}(n)$ if U reflects t times in every $G \in \mathcal{L}(n)$. Denote by $R(t, n)$ the length of a shortest t-reflecting sequence for $\mathcal{L}(n)$.

Theorem 1 reduces the length lower bounds for universal traversal sequences to those for t-reflecting sequences.

Theorem 1 (*[16]*).

1. *For every positive integer n, $U(2, 2n) \ge R(1, n)$.*
2. *For all positive integers d and n such that $d \ge 3$ and $16(d - 1)$ divides n,*

$$U(d, n) \ge \frac{d}{2} R\left(\frac{(d-2)n}{2} + 4, \frac{n}{16(d-1)}\right).$$

We refine Corollary 6 of [16] to obtain a lower bound on $R(t, n)$ in Lemma 3, by using the recurrence in Theorem 2.

Theorem 2 (*[16]*). *For all positive integers t, m, and n,*

$$R(t, mn) \ge R(R(t, m), n).$$

[1] An implementation of the algorithm is available upon request from the author.