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On the girth of the bipartite graph $D(k, q)$[☆]

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ABSTRACT

For integer $k \geq 2$ and prime power q, an algebraic bipartite graph $D(k, q)$ of girth at least $k + 4$ was introduced by Lazebnik and Ustimenko (1995). Füredi et al. (1995) shown that the girth of $D(k, q)$ is equal to $k+5$ if k is odd and q is a prime power of form $1+n(k+5)/2$ and, conjectured further that $D(k, q)$ has girth $k+5$ for all odd k and all $q \geq 4$. In this paper, we show that this conjecture is true when $(k + 5)/2$ is a power of the characteristic of $\mathbb{F}_q$.

1. Introduction

The graphs we consider in this paper are simple, i.e. undirected, without loops and multiple edges. For a graph G, its vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. The order of G is the number of vertices in $V(G)$. The degree of a vertex $v \in G$ is the number of the vertices that are adjacent to it. A graph is said r-regular if the degree of every vertex is equal to r. An automorphism of G means a bijection ϕ from $V(G)$ to itself such that $\{\phi(v), \phi(v')\}$ is an edge iff $\{v, v'\}$ is. G is said to be edge-transitive if for any two edges $\{v_1, v_1'\}, \{v_2, v_2'\}$ there is an automorphism ϕ of G such that $\{\phi(v_1), \phi(v_1')\} = \{v_2, v_2'\}$. A sequence $v_1v_2 \cdots v_n$ of vertices of G is called a path of length n if $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, 2, \ldots, n-1$ and $v_j \neq v_{j+2}$ for $j = 1, 2, \ldots, n-2$. A path $v_1v_2 \cdots v_n$ is called a cycle further if its length n is not smaller than 3 and $v_3v_4 \cdots v_nv_1v_2$ is still a path. If G contains at least one cycle, then the girth of G, denoted by $g(G)$, is the length of a shortest cycle in G. In the literature, graphs with large girth and a high degree of symmetry have been shown to be useful in different problems in extremal graph theory, finite geometry, coding theory, cryptography, communication networks and quantum computations (cf. [1–18]).

Let q be a prime power and $\mathbb{F}_q$ the finite field of q elements. For $k \geq 2$, in [7] Lazebnik and Ustimenko proposed a bipartite graph, denoted by $D(k, q)$, which is q-regular, edge-transitive and of large girth. The bipartite graph $D(k, q)$ can be equivalently described as follows (see [12]): The vertex sets $L(k)$ and $P(k)$ of $D(k, q)$ are two copies of $\mathbb{F}_q^k$ such that two vertices $(l_1, l_2, \ldots, l_k) \in L(k)$ and $(p_1, p_2, \ldots, p_k) \in P(k)$ are adjacent in $D(k, q)$ if and only if

$$l_2 + p_2 = p_1 l_1, \tag{1}$$
$$l_3 + p_3 = p_1 l_2, \tag{2}$$

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and, for $4 \le i \le k$,

$$l_i + p_i = \begin{cases} -p_{i-2}l_1, & \text{if } i \equiv 0 \text{ or } 1 \pmod 4, \\ p_1 l_{i-2}, & \text{if } i \equiv 2 \text{ or } 3 \pmod 4. \end{cases} \tag{3}$$

The construction of $D(k, q)$ was partially motivated by attempts to generalize the notion of the "affine part" of a generalized polygon. In fact, $D(2, q)$ and $D(3, q)$ (q odd) are exactly the affine parts of a regular generalized 3-gon and 4-gon, respectively [6,10]. In [7], some automorphisms of $D(k, q)$ were given and then the girth of $D(k, q)$ was shown to be at least $k + 4$. Later in [8], the graph $D(k, q)$ was shown to be disconnected in general. The number of components of $D(k, q)$ was determined further for $q \ge 3$ in [8,9,12]. The connectivity and the lower bound of the girth of $D(k, q)$ imply that the components of $D(k, q)$ provide the best-known asymptotic lower bound for the greatest number of edges in graphs of their order and girth (see [9,12]). It is of interest to know whether $D(k, q)$ has larger girth than the lower bound.

Since the length of any cycle in a bipartite graph must be an even integer, we see that the girth of $D(k, q)$ is at least $k + 5$ when k is odd. In [3], it was proved that the girth of $D(k, q)$ is equal to $k + 5$ when k is odd and q is a prime power of form $1 + n(k + 5)/2$. Furthermore, the following conjecture is proposed in [3]:

Conjecture A. *$D(k, q)$ has girth $k + 5$ for all odd k and all $q \ge 4$.*

We will show in this paper that Conjecture A is valid for a new infinite sequence of pairs (k, q).

This paper is arranged as follows. In Section 2 we propose an equivalent construction for the graph $D(k, q)$. For the proposed graph, a closed-form expression for some of its paths is given in Section 3 and some of its girth cycles are given in Section 4. In particular, for any prime p, the girth of $D(2p^s - 5, p^m)$ and that of $D(2p^s - 4, p^m)$ are proved in Section 4 to be equal to $2p^s$ if $p^s \ge 4$, where s and m are positive integers. An upper bound for the girth of $D(k, q)$ is also shown in Section 4.

2. An equivalent construction of $D(k, q)$

Clearly, if we set $l'_i = (-1)^{\lfloor i/4 \rfloor} l_i$ and $p'_i = (-1)^{\lfloor i/4 \rfloor} p_i$ for $i \ge 1$, then (1)–(3) can also be expressed as

$$l'_2 + p'_2 = p'_1 l'_1, \tag{4}$$

$$l'_3 + p'_3 = p'_1 l'_2, \tag{5}$$

and

$$l'_i + p'_i = \begin{cases} l'_1 p'_{i-2}, & \text{if } i \equiv 0 \text{ or } 1 \pmod 4, \\ p'_1 l'_{i-2}, & \text{if } i \equiv 2 \text{ or } 3 \pmod 4, \end{cases} \tag{6}$$

respectively, where $4 \le i \le k$. Below we will construct a new bipartite graph which is isomorphic to $D(k, q)$.

Let L be the set of infinite-dimensional vectors $(l_0, l_1, l_2, \ldots)$ over $\mathbb{F}_q$ with $l_1 = l_2$. Let R be the set of infinite-dimensional vectors $(r_0, r_1, r_2, \ldots)$ over $\mathbb{F}_q$ with $r_1 = 0$. For any vector in L or R, its second entry is indeed redundant. We will denote the vectors in L and R by $[l]$ and $\langle r \rangle$ respectively so that we can distinguish the origin of vectors in the union set $L \cup R$.

Let Λ_q be the bipartite graph with vertex set $V(\Lambda_q) = L \cup R$ and edge set $E(\Lambda_q) \subset L \times R$ such that $[l] = (l_0, l_1, \ldots) \in L$ and $\langle r \rangle = (r_0, r_1, \ldots) \in R$ are adjacent in Λ_q if and only if, for $i \ge 2$,

$$l_i + r_i = \begin{cases} r_0 l_{i-2} & \text{if } i \equiv 2, 3 \pmod 4, \\ l_0 r_{i-2} & \text{if } i \equiv 0, 1 \pmod 4. \end{cases} \tag{7}$$

For $k \ge 2$, let $\Lambda_{k,q}$ be the bipartite graph obtained from Λ_q by restricting vertices to their first $k + 1$ entries. From (4) to (6), we see that $\Lambda_{k,q}$ can be seen as obtained from $D(k, q)$ by just inserting a redundant entry between the first and the second entries of each vertex vector. Hence, $\Lambda_{k,q}$ is isomorphic to the bipartite graph $D(k, q)$. In particular, $\Lambda_{k,q}$ is q-regular and edge-transitive.

For any vector $a = (a_0, a_1, \ldots) \in \mathbb{F}_q^\infty$, let

$$\sigma a = (0, 0, a_0, a_1, \ldots),$$

$$\tau a = (a_2, a_3, \ldots),$$

$$\delta a = ((a_{4i}, a_{4i+1}, 0, 0)_{i=0,1,\ldots}),$$

where δ replaces with 0 the entries whose indices are of form $4i + 2$ or $4i + 3$. Clearly, σ, τ and δ are linear transformations on $\mathbb{F}_q^\infty$ satisfying

$$\tau\sigma = 1, \tag{8}$$

$$\delta^2 = \delta, \tag{9}$$

$$\sigma\delta + \delta\sigma = \sigma, \tag{10}$$

$$\tau\delta + \delta\tau = \tau, \tag{11}$$

$$\sigma\delta\tau = 1 - \delta, \tag{12}$$

where 1 denotes the identity transformation. Furthermore, we have the following lemma.