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Pairs of orthogonal countable ordinals

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ABSTRACT

We characterize pairs of orthogonal countable ordinals. Two ordinals α and β are orthogonal if there are two linear orders A and B on the same set V with order types α and β respectively such that the only maps preserving both orders are the constant maps and the identity map. We prove that if α and β are two countable ordinals, with $\alpha \leq \beta$, then α and β are orthogonal if and only if either $\omega + 1 \leq \alpha$ or $\alpha = \omega$ and $\beta < \omega\beta$.

1. Introduction

The following notion has been introduced by Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [5]:

Two orders P and Q on the same set are *orthogonal* if their only common order preserving maps are the identity map and the constant maps.

In this paper, we say that two ordinals α and β are *orthogonal* if there exist two linear orders A and B on the same set V with order types α and β respectively such that the only maps preserving both orders are the constant maps and the identity map. Let ω be the first infinite ordinal.

We prove:

Theorem 1. *If α and β are two countable ordinals, with $\alpha \leq \beta$, then α and β are orthogonal if and only if either $\omega + 1 \leq \alpha$ or $\alpha = \omega$ and $\beta < \omega\beta$.*

The proof of Theorem 1 will be done in Section 4, using the following result.

Theorem 2. *There are $2^{\aleph_0}$ linear orders L of order type ω on $\mathbb{N}$ such that L is orthogonal to the natural order on $\mathbb{N}$.*

This result will be given in Section 3, and follows from a simple construction which gives a bit more (see Corollary 2).

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Let us say few words about the history of this notion of orthogonality and our motivation.

The notion of orthogonality originates in the theory of clones. The first examples of pairs of orthogonal finite orders were given by Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [5]; those orders were in fact bipartite. More examples can be found in [6]. Nozaki, Miyakawa, Pogosyan and Rosenberg [19] investigated the existence of a linear order orthogonal to a given finite linear order. They observed that there is always one provided that the number of elements is not equal to three and proved:

Theorem 3 *([19]). The proportion $q(n)/n!$ of linear orders orthogonal to the natural order on $[n] := \{1, \dots, n\}$ goes to $e^{-2} = 0.1353\dots$ when n goes to infinity.*

Their counting argument was based on the fact that two linear orders on the same *finite* set are orthogonal if and only if they do not have a common nontrivial interval. The notion capturing the properties of intervals of a linear order was extended long ago to posets, graphs and binary structures and a decomposition theory has been developed (e.g. see [9,12,10,7]). One of the terms in use for this notion is *autonomous set*; structures with no nontrivial autonomous subset – the building blocks in the decomposition theory – are called *prime* (or *indecomposable*). With this terminology, the above fact can be expressed by saying that two linear orders $\mathcal{L}$ and $\mathcal{M}$ on the same finite set V are orthogonal if and only if the binary structure $B := (V, \mathcal{L}, \mathcal{M})$, that we call a *bichain*, is prime. This leads to results relating primality and orthogonality [22,25].

The notion of primality has reappeared in recent years under a quite different setting: a study of permutations motivated by the Stanley–Wilf conjecture, now settled by Marcus and Tardös [17]. This study, which developed in many papers, can be presented as follows: To a permutation σ on $[n]$ associate first the linear order $\leq_\sigma$ defined by $x \leq_\sigma y$ if $\sigma(x) \leq \sigma(y)$ for the natural order on $[n]$; next associate the bichain $B_\sigma := ([n], (\leq, \leq_\sigma))$. On the set $\mathfrak{S} := \cup_{n\in\mathbb{N}} \mathfrak{S}_n$ of all permutations, set $\sigma \leq \tau$ if B_σ is embeddable into B_τ. Say that a subset $\mathfrak{C}$ of $\mathfrak{S}$ is *hereditary* if $\sigma \leq \tau$ and $\tau \in \mathfrak{C}$ imply $\sigma \in \mathfrak{C}$. The goal is to evaluate the growth rate of the function $\varphi_\mathfrak{C}$ which counts for each integer n the numbers $\varphi_\mathfrak{C}(n)$ of permutations σ on $[n]$ which belong to $\mathfrak{C}$ (the Stanley–Wilf conjecture asserted that $\varphi_\mathfrak{C}$ is bounded by an exponential if $\mathfrak{C} \neq \mathfrak{S}$). For this purpose, simple permutations were introduced. A permutation σ is *simple* if $\leq_\sigma$ and the natural order $\leq$ on $[n]$ have no nontrivial interval in common. Arbitrary permutations being obtained by means of simple permutations, the enumeration of permutations belonging to a hereditary class of permutations can be then reduced to the enumeration of simple permutations belonging to that class. This fact was illustrated in many papers ([1,15], see also [4] for a survey on simple permutations and [2], where the asymptotic result mentioned in Theorem 3 is rediscovered). Notably, Albert and Atkinson [1] proved that the generating series $\sum_{n\in\mathbb{N}} \varphi_\mathfrak{C}(n)z^n$ is algebraic provided that $\mathfrak{C}$ contains only finitely many simple permutations. They asked for possible extensions of their result to hereditary sets containing infinitely many simple permutations.

Tools of the theory of relations provide easy ways to produce examples of hereditary sets containing infinitely many simple permutations (but not to answer the Albert–Atkinson question), see e.g. [20]. Let us say that a subset $\mathfrak{C}$ of $\mathfrak{S}$ is an *ideal* if it is non-empty, hereditary and *up-directed*, this last condition meaning that every pair $\sigma, \sigma' \in \mathfrak{C}$ has an upper bound $\tau \in \mathfrak{C}$. Let us call *age* of a bichain B the set $age(B) := \{\sigma \in \mathfrak{S} : B_\sigma \text{ is embeddable into } B\}$. Then, a subset $\mathfrak{C}$ of $\mathfrak{S}$ is an ideal if and only if $\mathfrak{C}$ is the age of some bichain B. Furthermore, an ideal $\mathfrak{C}$ is the age of a prime bichain if and only if every permutation belonging to $\mathfrak{C}$ is dominated by some simple permutation belonging to $\mathfrak{C}$ (these statements, which hold in the more general context of the theory of relations, are respectively due to Fraïssé [11] and Ille [13]). Because of these results, the study of ideals leads to the study of countable prime bichains. It is then natural to ask which are the possible pairs of order types of linear orders with this property. Now, it must be noticed that in the infinite case, primality and orthogonality no longer coincide. Thus, the next question is about pairs of orthogonal linear orders. In [24] it was proved that the chain of the rational numbers admits an orthogonal linear order of the same order type. Here we examine the case of countable well ordered chains.

2. Basic notations and results

Let V be a set. A *binary relation* on V is a subset ρ of the Cartesian product $V \times V$, but for convenience we write $x\rho y$ instead of $(x, y) \in \rho$. A map $f : V \to V$ *preserves* ρ if:

$$x\rho y \Rightarrow f(x)\rho f(y)$$

for all $x, y \in V$.

These two notions are enough to present our results. In order to prove them, we will need a bit more.

A *binary structure* is a pair $R := (V, (\rho_i)_{i\in I})$ where V is a set and each ρ_i is a binary relation on V. If F is a subset of V, the restriction of R to F is $R_{\restriction F} := (F, ((F \times F) \cap \rho_i)_{i\in I})$. If $R := (V, (\rho_i)_{i\in I})$ and $R' := (V', (\rho'_i)_{i\in I})$ are two binary structures, a *homomorphism* of R into R' is a map $f : V \to V'$ such that the implication

$$x\rho_i y \Rightarrow f(x)\rho'_i f(y) \tag{1}$$

holds for every $x, y \in V,\ i \in I$. If f is one-to-one and implication (1) above is a logical equivalence, this is an *embedding*. If $R = R'$, a homomorphism is an *endomorphism*. We will denote by $R \leq R'$ the fact that there is an embedding of R into R' and by $R \leq_{fin} R'$ the fact that $R_{\restriction(V\setminus F)} \leq R'$ for some finite subset F of V.